Oya, H.
Osaka J. Math.
55 (2018), 71–115

REPRESENTATIONS OF QUANTIZED COORDINATE ALGEBRAS VIA PBW-TYPE ELEMENTS

Hironori OYA

(Received February 1, 2016, revised November 4, 2016)

Abstract

Inspired by the work of Kuniba-Okado-Yamada, we study some tensor product representations of quantized coordinate algebras of symmetrizable Kac-Moody Lie algebras in terms of quantized enveloping algebras. As a consequence, we describe structures and properties of certain reducible representations of quantized coordinate algebras. This paper includes alternative proofs of Soibelman's tensor product theorem and Kuniba-Okado-Yamada's common structure theorem based on our direct calculation method using global bases.

Contents

1. Introduction

1.1. Backgrounds. Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. The quantized coordinate algebra $A_q[\mathfrak{g}]^+$ is the subspace of the dual space of the quantized enveloping algebra $U := U_q(\mathfrak{g})$ spanned by matrix coefficients of integrable modules of U. This subspace $A_q[\mathfrak{g}]^+$ has a Hopf algebra structure dual to that of U, and in particular $A_q[\mathfrak{g}]^+$ is a non-commutative algebra because of the non-cocommutativity of U. In this paper, we study representations of this non-commutative algebra $A_q[\mathfrak{g}]^+$.

In the early days of representation theory of $A_q[\mathfrak{g}]^+$, Soibelman et al. [27], [25] constructed the irreducible representations V_w of $A_q[\mathfrak{g}]^+$ corresponding to the elements w of the Weyl group W. These representations are infinite dimensional except for the case $w = e$. Through study of primitive ideals of $A_q[\mathfrak{g}]^+$ and highest weight theory for $A_q[\mathfrak{g}]^+$, Soibelman [25] has shown that the representation V_w can be constructed as a tensor product module. More precisely, there exist infinite dimensional irreducible modules $\{V_i := \bigoplus_{c\in\mathbb{Z}_{\geq 0}} \mathbb{Q}(q)|c\rangle_i\}_{i\in I}$ (I is the index set of simple roots of $\mathfrak{g}$) such that

2010 Mathematics Subject Classification. Primary 20G42; Secondary 17B37.

$$V_w \simeq V_{i_1} \otimes \cdots \otimes V_{i_l} (=: V_{\mathbf{i}})$$

for any reduced word $\mathbf{i} = (i_1, \ldots, i_l)$ of w. To put it another way, the $A_q[\mathfrak{g}]^+$-module $V_{\mathbf{i}}$ is irreducible for any reduced word $\mathbf{i}$ of w, and its isomorphism class does not depend on the choice of a reduced word of w. This result is called Soibelman's tensor product theorem. We fix an identification between V_w and $V_{\mathbf{i}}$.

By the way, we can construct an $A_q[\mathfrak{g}]^+$-module $V_{\mathbf{i}}$ for any sequence $\mathbf{i}$ of elements of I, which is reducible if $\mathbf{i}$ does not correspond to a reduced word of an element of W. To the best of my knowledge, however, little has been studied about the $A_q[\mathfrak{g}]^+$-modules $V_{\mathbf{i}}$ which do not correspond to reduced words. We also deal with some of such modules. See Subsection 1.2.

More recently, it has been found by Kuniba-Okado-Yamada [13] that there is a common structure in the positive part U^+ of U and the irreducible module V_{w_0} corresponding to the longest element w_0 of W. Let us explain it briefly. For a reduced word $\mathbf{i} = (i_1, \ldots, i_N)$ of w_0, the vector space V_{w_0} has a basis $\{|c_1\rangle_{i_1} \otimes \cdots \otimes |c_N\rangle_{i_N} (=: |\mathbf{c}\rangle_{\mathbf{i}})\}_{\mathbf{c} \in \mathbb{Z}_{\geq 0}^N}$, and the vector space U^+ has a basis $\{E_{\mathbf{i}}^{\mathbf{c}}\}_{\mathbf{c} \in \mathbb{Z}_{\geq 0}^N}$ known as a PBW-basis. See Definition 5.1 for the precise definition. Then the linear isomorphism $V_{w_0} \to U^+$ given by $|\mathbf{c}\rangle_{\mathbf{i}} \mapsto E_{\mathbf{i}}^{\mathbf{c}}$ does not depend on the choice of a reduced word $\mathbf{i}$. We call this fact Kuniba-Okado-Yamada's common structure theorem. This theorem suggests that the actions of $A_q[\mathfrak{g}]^+$ on V_{w_0} can be regarded as the ones on U^+. We will pursue this point of view in this paper. Kuniba-Okado-Yamada's proof is based on case-by-case computation in rank 2 cases. Subsequently, Tanisaki [26] proves this theorem through his realization of the module V_{w_0} as an induced module. Saito [23] also gives an alternative proof via an algebra homomorphism from the q-boson algebra to some localization of $A_q[\mathfrak{g}]^+$. In the present paper, we prove it by a direct calculation method using global bases. Actually, we consider the cases that $\mathfrak{g}$ is a symmetrizable Kac-Moody Lie algebra and an element of the Weyl group corresponding to an irreducible $A_q[\mathfrak{g}]^+$-module is not necessarily the longest element w_0. These are essentially Tanisaki's settings. Here we call these cases the generalized cases. See [26, Chapter 8] and Definition 5.14.

1.2. Results and Methods. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra. Our study on representations of $A_q[\mathfrak{g}]^+$ relies on the theory of global bases. This point is the main feature of our approach. Global bases which we use in this paper are bases of quantized enveloping algebras and their integrable highest (or extremal) weight modules [14], [4], [16], [7].

Global bases have many splendid properties. We collect some of them in Subsection 2.2. One of the most significant properties is the compatibility between the global bases of quantized enveloping algebras and those of their modules. In $A_q[\mathfrak{g}]^+$, we have the matrix coefficients $c^{\lambda}_{f_b, g_{b'}}$ determined by the elements of the lower global basis $g_{b'}$ and their dual basis f_b of the highest weight integrable U-module $V(\lambda)$ with highest weight λ. Through the compatibility property, we can write the direct calculation results of the actions of $c^{\lambda}_{f_b, g_{b'}}$ on the vector $|0\rangle_i$ in V_i ($i \in I$) using the structure constants associated with the global bases of quantized enveloping algebras (Proposition 3.10). Such direct calculation leads to one of our main theorems (Theorem 5.20), which describes a part of the action of $c^{\lambda}_{f_b, g_{b'}}$ on $|0\rangle_{\mathbf{i}}$ by the structure constants of comultiplication of U. Here, $\mathbf{i}$ is a reduced word of an element of W.

By the way, the existence of relations between the representation $V_{\mathbf{i}}$ and the structure of U^+ is also the claim of Kuniba-Okado-Yamada's common structure theorem. Indeed, we give an alternative proof of Kuniba-Okado-Yamada's common structure theorem together with Soibelman's tensor product theorem for the generalized cases as corollaries (Corollary 5.21, 5.23) of our theorem except for the irreducibility of V_w (Proposition 5.30). Roughly speaking, Corollary 5.21 allows us to compute the actions of the quantized coordinate algebras on V_w via the quantized enveloping algebras.

Next, we investigate the $A_q[\mathfrak{g}]^+$-module $\tilde{V}_w := V'_{w^{-1}} \otimes V_w$, where $V'_{w^{-1}}$ is the dual module of V_w in some sense. See Definition 5.14, 5.28 and Remark 5.17. The modules $\tilde{V}_w$ can be considered as tensor product type modules which do not correspond to reduced words. We study these modules $\tilde{V}_w$ through the terminology of quantized enveloping algebras, in the spirit of Kuniba-Okado-Yamada. Our bridge between quantized coordinate algebras and quantized enveloping algebras is an embedding $A_q[\mathfrak{g}]^+ \to \check{U}^{\leq 0} \otimes \check{U}^{\geq 0}$ of $\mathbb{Q}(q)$-algebras ([3, Chapter 9], Proposition 4.11), where $\check{U}^{\lesseqgtr 0}$ are variants of the Borel parts of U. As a consequence, we obtain the reasonable structures of the modules $\tilde{V}_w$.

Theorem (Theorem 5.32)**.** *The actions of the quantized coordinate algebras on $\tilde{V}_w$ are computed via the quantized enveloping algebras. In particular, the $A_q[\mathfrak{g}]^+$-module $\tilde{V}_w$ is decomposed into finite dimensional weight spaces $\tilde{V}_w = \bigoplus_{\alpha \in Q_+ \cap -w^{-1}Q_+} (\tilde{V}_w)_\alpha$ where*

$$(\tilde{V}_w)_\alpha := \{\tilde{\Lambda} \in \tilde{V}_w \mid c^{\lambda}_{f_\lambda, v_\lambda}.\tilde{\Lambda} = q^{(\lambda,\alpha)}\tilde{\Lambda} \text{ for all } \lambda \in P_+\}.$$

Here f_λ, v_λ are vectors of weight λ in $V(\lambda)^*$, $V(\lambda)$ respectively satisfying $\langle f_\lambda, v_\lambda \rangle = 1$. See Notation 2.1 *for the other standard notation in Lie theory. Moreover, the weight space $(\tilde{V}_w)_0$ is one dimensional, and $\tilde{V}_w$ is generated by this space.*

When w is not the unit element, any weight vector $0 \neq \tilde{\Lambda} \in (\tilde{V}_w)_\alpha$ generates a infinite dimensional submodule whose weight set is a subset of $(\alpha + Q_+) \cap -w^{-1}Q_+$, and its weight space with weight α is spanned by $\tilde{\Lambda}$.

Section 6 discusses the case that $\mathfrak{g}$ is of finite type. In subsection 6.1, we describe the relation between $\tilde{V}_{w_0}$ and $U^- \otimes U^+$ in the sense of the theorem above. In subsection 6.2, we show that, in some cases, the action of $A_q[\mathfrak{g}]^+$ on $\tilde{V}_w$ comes from that of the Drinfeld double $A_q[\mathfrak{g}]^+ \bowtie U'^{\mathrm{cop}}$ of $A_q[\mathfrak{g}]^+$ and U'^{cop}, where U'^{cop} is a variant of U (Theorem 6.7). This is one application of the theorem above, and a new result concerning tensor product modules $V_{\mathbf{i}}$ which do not correspond to reduced words. However we should remark that conceptual reasons why some $\tilde{V}_w$ admit compatible U'^{cop}-module structures are unclear now. It would be interesting to consider relations between these modules and the quantum double Bruhat cells studied, for example, in [1] and [2].

In the first draft [22] of this paper, we gave an alternative proof of the positivity (for ADE type) of the transition matrices from the lower global basis of U^+ to the PBW bases through the calculation in Theorem 5.20 with Lusztig's result [15, Theorem 11.5], and Soibelman, Kuniba-Okado-Yamada, Saito and Tanisaki's results. After submitting it to a preprint server, Yoshiyuki Kimura informed us that our proof of positivity can be simplified to the proof which do not require the representations of $A_q[\mathfrak{g}]^+$. See [22, Appendix]. Hence, in this version, we put this simplified proof in Subsection 5.1 as preliminaries and have changed the main application of Theorem 5.20 from the proof of positivity to the proof of Soibelman, Kuniba-Okado-Yamada, Saito and Tanisaki's results. The author wishes to express his thanks to Yoshiyuki Kimura.

The following are the basic notation and convention in this paper.

Notation 1.1. Let q be an indeterminate. For a $\mathbb{Q}(q)$-vector space V, set $V^* := \mathrm{Hom}_{\mathbb{Q}(q)}(V, \mathbb{Q}(q))$ and denote by $\langle\ ,\ \rangle$ the canonical pairing of V and V^*.

For a $\mathbb{Q}(q)$-algebra $\mathcal{A}$, we set $[a_1, a_2] := a_1a_2 - a_2a_1$ for $a_1, a_2 \in \mathcal{A}$. An $\mathcal{A}$-module V means a left $\mathcal{A}$-module. The action of $\mathcal{A}$ on V is often denoted by $a.v$ ($a \in \mathcal{A}, v \in V$). In this case, V^* is regarded as a right $\mathcal{A}$-module by $\langle f.a, v\rangle = \langle f, a.v\rangle$ ($f \in V^*, a \in \mathcal{A}, v \in V$).

For two letters i, j, the symbol δ_{ij} stands for the Kronecker delta.

2. Preliminaries

2.1. Quantized enveloping algebras.

Notation 2.1. Let $A = (a_{ij})_{i,j\in I}$ be a symmetrizable generalized Cartan matrix and $\mathfrak{g} := \mathfrak{g}(A)$ be the corresponding Kac-Moody Lie algebra (over $\mathbb{C}$). Let $\mathfrak{h}$ be its Cartan subalgebra, $\{\alpha_i\}_{i\in I}$ (resp. $\{\alpha_i^\vee\}_{i\in I}$) the set of simple roots (resp. coroots). Note that $a_{ij} = \langle\alpha_i^\vee, \alpha_j\rangle$. Set $Q := \sum_{i\in I}\mathbb{Z}\alpha_i$ (resp. $Q^\vee := \sum_{i\in I}\mathbb{Z}\alpha_i^\vee$) the root lattice (resp. the coroot lattice).

Take a $\mathbb{Z}$-lattice $P^\vee$ of $\mathfrak{h}$ such that

- $Q^\vee \subset P^\vee$,
- $Q \subset P := \{\lambda \in \mathfrak{h}^* \mid \langle\lambda, P^\vee\rangle \subset \mathbb{Z}\}$, and
- there exist elements $\{\varpi_i\}_{i\in I} \subset P$ such that $\langle\varpi_i, \alpha_j^\vee\rangle = \delta_{ij}$.

Fix elements $\{\varpi_i^\vee\}_{i\in I}$ such that $\langle\alpha_j, \varpi_i^\vee\rangle = \delta_{ji}$. (We do not assume $\{\varpi_i^\vee\}_{i\in I} \subset P^\vee$.) Define $Q_+ := \sum_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$. Set $P_+ := \{\lambda \in P \mid \langle\lambda, \alpha_i^\vee\rangle \geq 0 \text{ for all } i \in I\}$.

Denote by W the Weyl group of $\mathfrak{g}$, by $\{s_i\}_{i\in I}$ the simple reflections, by e the unit of W. For $w \in W$, denote by $I(w)$ the set of reduced words of w. (i.e. For $\mathbf{i} = (i_1, \ldots, i_l) \in I(w)$, $w = s_{i_1}\cdots s_{i_l}$ and l is the length of w.) Here we set $I(e) := \{\emptyset\}$. Take a W-invariant nondegenerate symmetric bilinear form $(\ ,\ )$ on $\mathfrak{h}^*$ such that $(\alpha_i, \alpha_i) = 2$ for all short simple roots α_i and $2(\alpha_i, \lambda)/(\alpha_i, \alpha_i) = \langle\alpha_i^\vee, \lambda\rangle$ for all $i \in I, \lambda \in \mathfrak{h}^*$. Write $\Delta_i := (\alpha_i, \alpha_i)/2$.

We define the partial order $\leq$ on P by $\mu \leq \lambda \Leftrightarrow \lambda - \mu \in Q_+$. For an element $\alpha := \sum_{i\in I} m_i\alpha_i \in Q$ ($m_i \in \mathbb{Z}$), we set $\mathrm{ht}\,\alpha := \sum_{i\in I} m_i$, called the height of α.

Notation 2.2. Set

$$q_i := q^{\Delta_i},\ [n] := \frac{q^n - q^{-n}}{q - q^{-1}} \text{ for } n \in \mathbb{Z},$$

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{cases} \dfrac{[n][n-1]\cdots[n-k+1]}{[k][k-1]\cdots[1]} & \text{if } n \in \mathbb{Z}, k \in \mathbb{Z}_{>0}, \\ 1 & \text{if } n \in \mathbb{Z}, k = 0, \end{cases}$$

$$[n]! := [n][n-1]\cdots[1] \text{ for } n \in \mathbb{Z}_{>0}, [0]! := 1.$$

Note that $[n], \begin{bmatrix} n \\ k \end{bmatrix} \in \mathbb{Z}[q^{\pm 1}]$ and $\begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{[n]!}{[k]![n-k]!}$ if $n \geq k \geq 0$. For a rational function $X \in \mathbb{Q}(q)$, we define X_i by the rational function obtained from X by substituting q by q_i ($i \in I$).

DEFINITION 2.3. The quantized enveloping algebra $U := U_q(\mathfrak{g})$ is the unital associative $\mathbb{Q}(q)$-algebra defined by the generators

$$E_i,\ F_i\ (i \in I),\ K_h\ (h \in P^\vee),$$

and the relations (i)-(iv) below:

(i) $K_0 = 1,\ K_h K_{h'} = K_{h+h'}$ for $h, h' \in P^\vee$,

(ii) $K_h E_i = q^{\langle h,\alpha_i\rangle} E_i K_h, K_h F_i = q^{-\langle h,\alpha_i\rangle} F_i K_h$ for $h \in P^\vee, i \in I$,

(iii) $[E_i, F_j] = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$ where $K_i := K_{\Delta_i \alpha_i^\vee}$,

(iv) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k X_i^{(k)} X_j X_i^{(1-a_{ij}-k)} = 0$ for $i, j \in I$ with $i \neq j$, where $X_i^{(n)} := X_i^n/[n]_i!, X = E$ and F.

For $\alpha = \sum_{i\in I} m_i\alpha_i \in Q$ ($m_i \in \mathbb{Z}$), we set $K_\alpha := \prod_{i\in I} K_i^{m_i}$. The $\mathbb{Q}(q)$-subalgebra of U generated by $\{E_i\}_{i\in I}$ (resp. $\{F_i\}_{i\in I}$, $\{K_h\}_{h\in P^\vee}$, $\{E_i, K_h\}_{i\in I,h\in P^\vee}$, $\{F_i, K_h\}_{i\in I,h\in P^\vee}$) is denoted by U^+ (resp. U^-, U^0, $U^{\geq 0}$, $U^{\leq 0}$).

For $\alpha \in Q$, write $U_\alpha := \{X \in U \mid K_h X K_{-h} = q^{\langle\alpha,h\rangle} X$ for all $h \in P^\vee\}$. The elements of U_α are called homogeneous. For a homogeneous element $u \in U_\alpha$, we set $\mathrm{wt}\, u = \alpha$. The subspaces $U_\alpha^\pm$ etc. are defined similarly.

The algebra U has a Hopf algebra structure given by the following comultiplication Δ, counit ε and antipode S:

$$\begin{aligned}
\Delta(E_i) &= E_i \otimes K_i^{-1} + 1 \otimes E_i, & \varepsilon(E_i) &= 0, & S(E_i) &= -E_i K_i,\\
\Delta(F_i) &= F_i \otimes 1 + K_i \otimes F_i, & \varepsilon(F_i) &= 0, & S(F_i) &= -K_i^{-1} F_i,\\
\Delta(K_h) &= K_h \otimes K_h, & \varepsilon(K_h) &= 1, & S(K_h) &= K_{-h},
\end{aligned}$$

for $i \in I, h \in P^\vee$.

DEFINITION 2.4. We define the $\mathbb{Q}(q)$-algebra, anti-coalgebra involution $\omega : U \to U$ by

$$\omega(E_i) = F_i,\quad \omega(F_i) = E_i,\quad \omega(K_h) = K_{-h},$$

for $i \in I, h \in P^\vee$. We define the $\mathbb{Q}(q)$-algebra anti-involutions $*, \psi : U \to U$ by

$$\begin{aligned}
*(E_i) &= E_i, & *(F_i) &= F_i, & *(K_h) &= K_{-h},\\
\psi(E_i) &= q_i^{-1} K_i^{-1} F_i, & \psi(F_i) &= q_i E_i K_i, & \psi(K_h) &= K_h,
\end{aligned}$$

for $i \in I, h \in P^\vee$. Note that ψ is a $\mathbb{Q}(q)$-coalgebra involution. We define the $\mathbb{Q}$-algebra involution $\overline{(\cdot)} : U \to U$ by

$$\overline{E_i} = E_i,\quad \overline{F_i} = F_i,\quad \overline{K_h} = K_{-h},\quad \overline{q} = q^{-1},$$

for $i \in I, h \in P^\vee$.

DEFINITION 2.5. For $i \in I$, define the $\mathbb{Q}(q)$-linear maps $e_i', {}_ie' : U^- \to U^-$ by

$$\begin{aligned}
&e_i'(Y_1Y_2) = e_i'(Y_1)Y_2 + q_i^{\langle \mathrm{wt}\, Y_1,\alpha_i^\vee\rangle} Y_1 e_i'(Y_2) \text{ and } e_i'(F_j) = \delta_{ij},\\
&{}_ie'(Y_1Y_2) = q_i^{\langle \mathrm{wt}\, Y_2,\alpha_i^\vee\rangle}\, {}_ie'(Y_1)Y_2 + Y_1\, {}_ie'(Y_2) \text{ and } {}_ie'(F_j) = \delta_{ij},
\end{aligned}$$

for $j \in I$ and homogeneous elements $Y_1, Y_2 \in U^-$.

For $i \in I$, define the $\mathbb{Q}(q)$-linear maps $f_i', {}_if' : U^+ \to U^+$ by

$$f_i'(X_1X_2) = f_i'(X_1)X_2 + q_i^{\langle \mathrm{wt}\, X_1, \alpha_i^\vee \rangle} X_1 f_i'(X_2) \text{ and } f_i'(E_j) = \delta_{ij},$$
$${}_if'(X_1X_2) = q_i^{\langle \mathrm{wt}\, X_2, \alpha_i^\vee \rangle}\, {}_if'(X_1)X_2 + X_1\, {}_if'(X_2) \text{ and } {}_if'(E_j) = \delta_{ij},$$

for $j \in I$ and homogeneous elements $X_1, X_2 \in U^+$.

We have $* \circ {}_ie' \circ *|_{U^-} = e_i'$, $* \circ {}_if' \circ *|_{U^+} = f_i'$.

For any homogeneous elements $X \in U^+$, $Y \in U^-$ and $p \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned}
\Delta(X) &= E_i^{(p)} \otimes q_i^{-\frac{1}{2}p(p-1)} K_{-p\alpha_i} (f_i')^p(X) + \sum_{\substack{X' \in U^+ \text{ homogeneous,} \\ \mathrm{wt}\, X' \neq p\alpha_i}} X' \otimes K_{-\mathrm{wt}\, X'} X'', \\
&= q_i^{-\frac{1}{2}p(p-1)} ({}_if')^p(X) \otimes K_{-\mathrm{wt}\, X + p\alpha_i} E_i^{(p)} + \sum_{\substack{X'' \in U^+ \text{ homogeneous,} \\ \mathrm{wt}\, x'' \neq p\alpha_i}} X' \otimes K_{-\mathrm{wt}\, X'} X'', \\
\Delta(Y) &= F_i^{(p)} K_{-\mathrm{wt}\, Y - p\alpha_i} \otimes q_i^{\frac{1}{2}p(p-1)} (e_i')^p(Y) + \sum_{\substack{Y' \in U^- \text{ homogeneous,} \\ \mathrm{wt}\, Y' \neq -p\alpha_i}} Y' K_{-\mathrm{wt}\, Y''} \otimes Y'', \\
&= q_i^{\frac{1}{2}p(p-1)} ({}_ie')^p(Y) K_{p\alpha_i} \otimes F_i^{(p)} + \sum_{\substack{Y'' \in U^- \text{ homogeneous,} \\ \mathrm{wt}\, Y'' \neq -p\alpha_i}} Y' K_{-\mathrm{wt}\, Y''} \otimes Y''.
\end{aligned}$$

For homogeneous elements $X \in U^+$, $Y \in U^-$ and $p \in \mathbb{Z}_{\geq 0}$, we have

$$(f_i')^p(X) = q_i^{p\langle \mathrm{wt}\, X, \alpha_i^\vee \rangle - p(p+1)} \overline{{}_if'^p(\overline{X})}, \quad (e_i')^p(Y) = q_i^{p\langle \mathrm{wt}\, Y, \alpha_i^\vee \rangle + p(p+1)} \overline{{}_ie'^p(\overline{Y})}.$$

For $X \in U^+$ and $Y \in U^-$, we have

$$[F_i, X] = -\frac{{}_if'(X)K_i - K_i^{-1} f_i'(X)}{q_i - q_i^{-1}}, \quad [E_i, Y] = \frac{{}_ie'(Y)K_i - K_i^{-1} e_i'(Y)}{q_i - q_i^{-1}} \tag{2.1}$$

in U. See [18, Chapter 1, 3] for details. (Our definition of comultiplication is slightly different from the one in the reference [18].)

Definition 2.6. The modified quantized enveloping algebra $\dot{U} = \bigoplus_{\lambda, \lambda' \in P} {}_\lambda \dot{U}_{\lambda'}$ is defined as in [18, Chapter 23]. Here ${}_\lambda \dot{U}_{\lambda'} := U / (\sum_{h \in P^\vee} (K_h - q^{\langle \lambda, h \rangle}) U + \sum_{h \in P^\vee} U (K_h - q^{\langle \lambda', h \rangle}))$. We follow the notation of [18, Chapter 23] except for $a_\lambda := \pi_{\lambda, \lambda}(1)$ ($\lambda \in P$). Here $\pi_{\lambda, \lambda'}$ denotes the canonical projection $U \to {}_\lambda \dot{U}_{\lambda'}$ for $\lambda, \lambda' \in P$. These elements satisfy $a_\lambda a_{\lambda'} = \delta_{\lambda, \lambda'} a_\lambda$ and ${}_\lambda \dot{U}_{\lambda'} = a_\lambda \dot{U} a_{\lambda'}$.

By abuse of notation, the comultiplication $\dot{U} \to \prod_{\lambda_1, \lambda_1', \lambda_2, \lambda_2'} {}_{\lambda_1} \dot{U}_{\lambda_1'} \otimes {}_{\lambda_2} \dot{U}_{\lambda_2'}$ of $\dot{U}$ will also be denoted by Δ.

The maps $\omega, *, \psi, \overline{(\cdot)}$ are naturally regarded as the ones on $\dot{U}$. In particular, $\overline{a_\lambda} = \psi(a_\lambda) = a_\lambda$ and $\omega(a_\lambda) = *(a_\lambda) = a_{-\lambda}$.

Definition 2.7. Let V be a left (resp. right) U-module. For any $\lambda \in P$, we set

$$V_\lambda := \{ v \in V \mid K_h.v = q^{\langle h, \lambda \rangle} v \text{ (resp. } v.K_h = q^{\langle h, \lambda \rangle} v) \text{ for all } h \in P^\vee \}.$$

Let $\mathcal{C}$ (resp. $\mathcal{C}^{\mathrm{r}}$) be the category of left (resp. right) U-modules V with the weight space decomposition $V = \bigoplus_{\lambda \in P} V_\lambda$. The category $\mathcal{C}$ (resp. $\mathcal{C}^{\mathrm{r}}$) is closed under the tensor product of U-modules.

A left (resp. right) $\dot{U}$-module V' is said to be unital if it suffices the following conditions:

- For any $v \in V'$, we have $a_\lambda.v = 0$ (resp. $v.a_\lambda = 0$) except for finitely many $\lambda \in P$.
- For any $v' \in M'$, we have $\sum_{\lambda \in P} a_\lambda.v' = v'$ (resp. $\sum_{\lambda\in P} v'.a_\lambda = v'$).

Let C' (resp. C'^{r}) be the category of left (resp. right) unital $\dot{U}$-modules. The tensor product $V' \otimes W'$ of unital $\dot{U}$-modules V', W' has the unital $\dot{U}$-module structure via Δ.

Any object $V = \bigoplus_{\lambda\in P} V_\lambda$ of C can be regarded as an object of C'; for $\lambda, \lambda' \in P, X \in U_{\lambda-\lambda'}, v = \sum_{\mu\in P} v_\mu \in V$ with $v_\mu \in V_\mu$, set $\pi_{\lambda,\lambda'}(X).v := X.v_{\lambda'}$.

Conversely, any object V' of C' can be regarded as an object of C; for $X \in U$ and $v' \in V'$, we set $X.v' = \sum_{\lambda\in P}(X.a_\lambda).v'$. Then, $V' = \bigoplus_{\lambda\in P} V'_\lambda$ and $V'_\lambda = a_\lambda.V'$.

These correspondences give rise to the isomorphism of tensor categories between C and C'. Similarly, C^{r} is isomorphic to C'^{r}.

From now on, we do not distinguish the category C (resp. C^{r}) from the category C' (resp. C'^{r}).

We say that V is integrable if

- V is an object of C (or C^{r}), and
- the actions of E_i and F_i on V are locally nilpotent for all $i \in I$.

The tensor product of integrable modules is also integrable.

Let V be an integrable U-module. A vector $v \in V$ of weight $\lambda \in P$ is said to be extremal if there exists a set of vectors $\{v_w\}_{w\in W}$ such that

- $v_e = v$,
- if $\langle \alpha_i^\vee, w\lambda\rangle \geq 0$, then $E_i.v_w = 0$ and $F_i^{(\langle \alpha_i^\vee, w\lambda\rangle)}.v_w = v_{s_i w}$,
- if $\langle \alpha_i^\vee, w\lambda\rangle \leq 0$, then $F_i.v_w = 0$ and $E_i^{(-\langle \alpha_i^\vee, w\lambda\rangle)}.v_w = v_{s_i w}$.

For $\lambda \in P$, we denote by $V(\lambda)$ the U-module generated by v_λ with the defining relation that v_λ is an extremal vector of weight λ. The module $V(\lambda)$ is integrable.

Remark 2.8. For $\lambda \in P$ and $i \in I$, we have the isomorphism $V(\lambda) \to V(s_i\lambda)$ of U-modules given by

$$v_\lambda \mapsto X_i^{(|\langle \lambda, \alpha_i^\vee\rangle|)} v_{s_i\lambda} \text{ where } X := \begin{cases} E & \text{if } \langle \lambda, \alpha_i^\vee\rangle \geq 0 \\ F & \text{if } \langle \lambda, \alpha_i^\vee\rangle \leq 0. \end{cases}$$

Definition 2.9. Let V be an object of C (resp. C'). For $f \in V^*$ and $v \in V$, define an element $c_{f,v}^V \in U^*$ (resp. $\in \dot{U}^*$) by

$$X \mapsto \langle f, X.v\rangle (X \in U \text{ resp. } X \in \dot{U}).$$

Let MC_C (resp. $MC_{C'}$) be the subspace of U^* (resp. $\dot{U}^*$) spanned by all elements of the form $c_{f,v}^V$ (V's are objects of C resp. C'). Then, MC_C and $MC_{C'}$ are $\mathbb{Q}(q)$-algebras whose multiplications are defined as the dual of the comultiplications Δ. Then the $\mathbb{Q}(q)$-linear map $MC_C \to MC_{C'}, c_{f,v}^V \mapsto c_{f,v}^V$ is an isomorphism of $\mathbb{Q}(q)$-algebras. Hence we will identify $c_{f,v}^V \in U^*$ with $c_{f,v}^V \in \dot{U}^*$.

For $\lambda \in P$, the element $c_{f,v}^{V(\lambda)}$ will be briefly denoted by $c_{f,v}^\lambda$.

Definition 2.10. Let V be an integrable U-module. For $i \in I$ and $\epsilon \in \{\pm 1\}$, there exist $\mathbb{Q}(q)$-linear automorphisms $T'_{i,\epsilon}, T''_{i,\epsilon}$ on V and the $\mathbb{Q}(q)$-algebra automorphisms $T'_{i,\epsilon}, T''_{i,\epsilon}$ on U defined as in [18, Chapter 5, 37]. It is known that these maps satisfy the following properties

[18]:

- $T'_{i,\epsilon} = (T''_{i,-\epsilon})^{-1}$ on V and U.
- $T^{\flat}_{i,\epsilon}(X.v) = T^{\flat}_{i,\epsilon}(X).T^{\flat}_{i,\epsilon}(v)$ for $v \in V$ and $X \in U$ ($\flat \in \{\,', \,''\}$).
- The maps $\{T^{\flat}_{i,\epsilon}\}_{i\in I}$ satisfy the braid relations. Hence, for $w \in W$, the map $T^{\flat}_{w,\epsilon} := T^{\flat}_{i_1,\epsilon} \cdots T^{\flat}_{i_l,\epsilon}$ with $\mathbf{i} = (i_1, \ldots, i_l) \in I(w)$ is well-defined on V and U ($\flat \in \{\,', \,''\}$).
- $\omega \circ T'_{i,\epsilon} \circ \omega = T''_{i,\epsilon}$, $* \circ T'_{i,\epsilon} \circ * = T''_{i,-\epsilon}$ and $\overline{(\cdot)} \circ T^{\flat}_{i,\epsilon} \circ \overline{(\cdot)} = T^{\flat}_{i,-\epsilon}$ on U ($\flat \in \{\,', \,''\}$).

Proposition 2.11 ([18, Proposition 5.3.4]). *Let V and V' be integrable U-modules. Then,*

$$T''_{i,-1}\Big(\sum_{k\geq 0} q_i^{\frac{k(k-1)}{2}} \prod_{s=1}^{k} (q_i^s - q_i^{-s}) E_i^{(k)} \otimes F_i^{(k)}.\tilde{v}\Big) = (T''_{i,-1} \otimes T''_{i,-1})(\tilde{v}),$$

for any $i \in I$ and $\tilde{v} \in V \otimes V'$.

2.2. Global bases.

Definition 2.12. We refer to [5] for the definition of the category of Kashiwara crystals $(B, \mathrm{wt}, \{\varepsilon_i\}_{i\in I}, \{\varphi_i\}_{i\in I}, \{\tilde{e}_i\}_{i\in I}, \{\tilde{f}_i\}_{i\in I})$ associated with $(P, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$. Recall that $\varphi_i(b) = \varepsilon_i(b) + \langle \alpha_i^\vee, \mathrm{wt}\, b\rangle$ for $b \in B$, which is one of their axioms. A morphism $\Psi : B_1 \to B_2$ of crystals is said to be strict if Ψ commutes with the Kashiwara operators $\tilde{e}_i$'s and $\tilde{f}_i$'s ($i \in I$), and to be an embedding if the associated map $\Psi : B_1 \coprod \{0\} \to B_2 \coprod \{0\}$ is injective.

For crystals B_1 and B_2, the tensor product $B_1 \otimes B_2$ of B_1 and B_2 is defined as follows [5]:

(i) $B_1 \otimes B_2 = B_1 \times B_2$ as a set, (An element $(b_1, b_2) \in B_1 \otimes B_2$ will be denoted by $b_1 \otimes b_2$)
(ii) $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}\, b_1 + \mathrm{wt}\, b_2$,
(iii) $\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \alpha_i^\vee, \mathrm{wt}\, b_1\rangle\}$,
(iv) $\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_1) + \langle \alpha_i^\vee, \mathrm{wt}\, b_2\rangle, \varphi_i(b_2)\}$,
(v) $\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{otherwise,} \end{cases}$
(vi) $\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{otherwise,} \end{cases}$

where we set $0 \otimes b = b \otimes 0 = 0$.

For a crystal B, a new crystal $B^\omega := \{b^\omega \mid b \in B\}$ is defined as $\mathrm{wt}\, b^\omega = -\mathrm{wt}\, b, \varepsilon_i(b^\omega) = \varphi_i(b), \varphi_i(b^\omega) = \varepsilon_i(b), \tilde{e}_i(b^\omega) = (\tilde{f}_i b)^\omega$ and $\tilde{f}_i(b^\omega) = (\tilde{e}_i b)^\omega$.

Definition 2.13. For $\lambda \in P$, we define the crystal $T_\lambda = \{t_\lambda\}$ by $\mathrm{wt}\, t_\lambda = \lambda, \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$, and $\tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0$.

For $i \in I$, the crystal $B_i := \{b_i(n) \mid n \in \mathbb{Z}\}$ is defined as

$$\mathrm{wt}\, b_i(n) = n\alpha_i, \varepsilon_j(b_i(n)) = \begin{cases} -n & \text{if } j = i, \\ -\infty & \text{if } j \neq i, \end{cases} \varphi_j(b_i(n)) = \begin{cases} n & \text{if } j = i, \\ -\infty & \text{if } j \neq i, \end{cases}$$

$$\tilde{e}_i b_i(n) = b_i(n+1) \text{ and } \tilde{f}_i b_i(n) = b_i(n-1).$$

We denote by $B(\infty)$ (resp. $B(\dot{U})$, $B(\dot{U}a_\lambda)$, $B(\lambda)(\lambda \in P)$) the crystal associated with U^- (resp. $\dot{U}$, $\dot{U}a_\lambda$, $V(\lambda)$). See [4], [7] for the precise definitions. The unique element of $B(\infty)$ with weight 0 will be denoted by $\tilde{b}_\infty$. Set $B(-\infty) := B(\infty)^\omega$. Then, the crystal $B(-\infty)$ is isomorphic to the crystal associated with U^+. We have

$$B(\dot{U}) = \coprod_{\lambda\in P} B(\dot{U}a_\lambda) \text{ and } B(\dot{U}a_\lambda) \simeq B(\infty)\otimes T_\lambda \otimes B(-\infty)(\lambda \in P)$$

as crystals [7, Theorem 3.1.1]. Henceforth, we regard the elements of $B(\infty)\otimes T_\lambda\otimes B(-\infty)$ as those of $B(\dot{U}a_\lambda)$. We refer to [8, Appendix B] for a dictionary of calculations on this crystal.

The anti-involution $*$ induces the bijections on $B(\infty)$ and $B(\dot{U})$. ([5, Theorem 2.1.1], [7, Theorem 4.3.2]) Moreover, these bijections give the new crystal structures on $B(\infty)$ and $B(\dot{U})$, defined by the maps $\mathrm{wt}^* := \mathrm{wt}\circ *, \varepsilon_i^* := \varepsilon_i \circ *, \varphi_i^* := \varphi_i\circ *, \tilde{e}_i^* := *\circ\tilde{e}_i\circ *, \tilde{f}_i^* := *\circ\tilde{f}_i\circ *$. Note that $\mathrm{wt}^* = \mathrm{wt}$ for $B(\infty)$.

Proposition 2.14 ([5, Theorem 2.2.1])**.** *For $i \in I$, these exists a unique strict embedding $\Psi_i : B(\infty) \to B(\infty)\otimes B_i$ of crystals given by $\tilde{b}_\infty \mapsto \tilde{b}_\infty \otimes b_i(0)$.*

Moreover, $\Psi_i(\tilde{b}) = \tilde{e}_i^{*\varepsilon_i^*(\tilde{b})}\tilde{b}\otimes b_i(-\varepsilon_i^*(\tilde{b}))$ for all $\tilde{b} \in B(\infty)$.

Corollary 2.15 ([24, Lemma 3.4.6])**.** *For any $\tilde{b} \in B(\infty)$, we have*

$$\varepsilon_i(\tilde{b}) + \varphi_i^*(\tilde{b}) = \varepsilon_i^*(\tilde{b}) + \varphi_i(\tilde{b}) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \varepsilon_i((\tilde{f}_i^*)^k\tilde{b}) = \varepsilon_i(\tilde{b})\}.$$

In particular, $\varepsilon_i(\tilde{b}) + \varphi_i^*(\tilde{b}) = \varepsilon_i^*(\tilde{b}) + \varphi_i(\tilde{b}) \geq 0$.

Corollary 2.16. *Let $i \in I$ and $\tilde{b} \in B(\infty)$. Suppose that $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ are any composition of the operators $\tilde{e}_i, \tilde{f}_i, \tilde{e}_i^*$ and $\tilde{f}_i^*$ such that $\tilde{\kappa}_1(\tilde{b}) \neq 0$ and $\tilde{\kappa}_2(\tilde{b}) \neq 0$.*

Then, we have $\tilde{\kappa}_1(\tilde{b}) = \tilde{\kappa}_2(\tilde{b})$ if and only if

$$\langle \mathrm{wt}\,\tilde{\kappa}_1(\tilde{b}), \alpha_i^\vee\rangle = \langle \mathrm{wt}\,\tilde{\kappa}_2(\tilde{b}), \alpha_i^\vee\rangle \text{ and } \varepsilon_i^*(\tilde{\kappa}_1(\tilde{b})) = \varepsilon_i^*(\tilde{\kappa}_2(\tilde{b})).$$

Proof. Write $\Psi_i(\tilde{b}) = \tilde{b}_0 \otimes b_i(m_0), \Psi_i(\tilde{\kappa}_1(\tilde{b})) = \tilde{b}_1\otimes b_i(m_1), \Psi_i(\tilde{\kappa}_2(\tilde{b})) = \tilde{b}_2\otimes b_i(m_2)$ and

$$B_0^{(i)} := \left(\{\tilde{e}_i^p\tilde{b}_0 \mid p \in \mathbb{Z}_{\geq 0}\} \cup \{\tilde{f}_i^p\tilde{b}_0 \mid p\in\mathbb{Z}_{\geq 0}\}\right) \setminus \{0\}.$$

Then, by the definition of the crystal structure on $B(\infty)\otimes B_i$, we have $\tilde{b}_1, \tilde{b}_2 \in B_0^{(i)}$. Therefore, we have

$$\begin{aligned}\tilde{\kappa}_1(\tilde{b}) = \tilde{\kappa}_2(\tilde{b}) &\Leftrightarrow \Psi_i(\tilde{\kappa}_1(\tilde{b})) = \Psi_i(\tilde{\kappa}_2(\tilde{b}))(\Leftrightarrow \tilde{b}_1 = \tilde{b}_2 \text{ and } m_1 = m_2),\\ &\Leftrightarrow \langle \mathrm{wt}\,\tilde{b}_1, \alpha_i^\vee\rangle = \langle \mathrm{wt}\,\tilde{b}_2,\alpha_i^\vee\rangle \text{ and } \varepsilon_i^*(\tilde{\kappa}_1(\tilde{b})) = \varepsilon_i^*(\tilde{\kappa}_2(\tilde{b})),\\ &\Leftrightarrow \langle \mathrm{wt}\,\tilde{\kappa}_1(\tilde{b}),\alpha_i^\vee\rangle = \langle \mathrm{wt}\,\tilde{\kappa}_2(\tilde{b}),\alpha_i^\vee\rangle \text{ and } \varepsilon_i^*(\tilde{\kappa}_1(\tilde{b})) = \varepsilon_i^*(\tilde{\kappa}_2(\tilde{b})).\end{aligned}$$

□

Proposition 2.17 ([7, Corollary 4.3.3])**.** *For $\tilde{b}_1 \in B(\infty), \tilde{b}_2 \in B(-\infty)$ and $\lambda \in P$, we have*

$$*(\tilde{b}_1\otimes t_\lambda\otimes\tilde{b}_2) = *(\tilde{b}_1)\otimes t_{-\lambda-\mathrm{wt}\,\tilde{b}_1-\mathrm{wt}\,\tilde{b}_2}\otimes *(\tilde{b}_2)$$

as elements of $B(\dot{U})$. In particular, $\mathrm{wt}^*\,\dot{b} = -\lambda$ *for $\dot{b} \in B(\dot{U}a_\lambda)$.*

Definition 2.18. For a subset $J \subset I$, we denote by U_J the subalgebra of U generated by $\{E_j, F_j, K_{\alpha_i^\vee}\}_{j\in J}$.

A crystal B is called normal if, for any subset $J \subset I$ such that $(a_{j,k})_{j,k\in J}$ is of finite type, B is isomorphic to a crystal of an integrable U_J-module as a crystal associated with the data $(P|_{\sum_{j\in J}\mathbb{Z}\alpha_j^\vee}, \{\alpha_j\}_{j\in J}, \{\alpha_j^\vee\}_{j\in J})$.

For a normal crystal B, $b \in B$ and $i \in I$, we have

$$\varepsilon_i(b) = \max\{m \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^m b \neq 0\} \text{ and } \varphi_i(b) = \max\{m \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^m b \neq 0\}.$$

For a normal crystal B and $i \in I$, we define the bijection $S_i : B \to B$ by

$$S_i b = \begin{cases} \tilde{f}_i^{\langle \alpha_i^\vee, \mathrm{wt}\, b\rangle} b & \text{if } \langle \alpha_i^\vee, \mathrm{wt}\, b\rangle \geq 0 \\ \tilde{e}_i^{-\langle \alpha_i^\vee, \mathrm{wt}\, b\rangle} b & \text{if } \langle \alpha_i^\vee, \mathrm{wt}\, b\rangle \leq 0 \end{cases} \quad \text{for } b \in B.$$

Note that $S_i^2 = \mathrm{id}$. The maps $\{S_i\}_{i \in I}$ satisfy the braid relation ([7, Theorem 7.2.2]). Hence, the map $S_w := S_{i_1} \cdots S_{i_l}$ is well-defined for any $w = s_{i_1} \cdots s_{i_l} \in W$.

For a normal crystal B, an element $b \in B$ is called extremal if

- $\tilde{e}_i S_w b = 0$ for all $w \in W$ with $\langle \alpha_i^\vee, w\, \mathrm{wt}\, b\rangle \geq 0$, and
- $\tilde{f}_i S_w b = 0$ for all $w \in W$ with $\langle \alpha_i^\vee, w\, \mathrm{wt}\, b\rangle \leq 0$.

Remark 2.19. We have the following properties:

- The crystals $T_\lambda (\lambda \in P)$, $B_i (i \in I)$ and $B(\pm\infty)$ are not normal.
- The crystals $B(\dot{U})$ and $B(\lambda)(\lambda \in P)$ are normal. ([16], [7])
- For $\lambda \in P$, the crystal $B(\lambda)$ can be identified with the subcrystal $\{\dot{b} \in B(\dot{U}a_\lambda) \mid *\dot{b}$ is extremal$\}$ of $B(\dot{U}a_\lambda)$. ([7, Proposition 8.2.2])
- For $\lambda \in P$ and $i \in I$, there exists the isomorphism $B(\lambda) \to B(s_i\lambda)$ of crystals given by $b \mapsto (S_i(b^*))^* (=: S_i^* b)$. ([7, Proposition 8.2.2])

Notation 2.20. Let us denote by $\{G^-(\tilde{b})\}_{\tilde{b} \in B(\infty)}$ (resp. $\{G^+(\tilde{b})\}_{\tilde{b} \in B(-\infty)}$, $\{G(\dot{b})\}_{\dot{b} \in B(\dot{U})}$, $\{g_b\}_{b \in B(\lambda)}$) the lower global basis of U^- (resp. $U^+, \dot{U}, V(\lambda)(\lambda \in P)$). We refer to [4], [16], [7] for the definitions of them.

Remark 2.21. The following are well-known properties of the lower global bases.

- $\omega(G^-(\tilde{b})) = G^+(\tilde{b}^\omega)$ for $\tilde{b} \in B(\infty)$.
- $\overline{G^\mp(\tilde{b})} = G^\mp(\tilde{b}), \overline{G(\dot{b})} = G(\dot{b})$ for $\tilde{b} \in B(\pm\infty)$, $\dot{b} \in B(\dot{U})$. ([4], [16])
- $*G^\mp(\tilde{b}) = G^\mp(*\tilde{b}), *G(\dot{b}) = G(*\dot{b})$ for $\tilde{b} \in B(\pm\infty)$, $\dot{b} \in B(\dot{U})$. ([5], [7])
- For $\lambda \in P_+$, we have $\{G^\mp(\tilde{b}).v_{\pm\lambda}\}_{\tilde{b} \in B(\pm\infty)} \setminus \{0\} = \{g_b\}_{b \in B(\pm\lambda)}$. Moreover, the maps $B(\lambda) \to B(\infty) \otimes T_\lambda, b \mapsto \tilde{b} \otimes t_\lambda$ with $g_b = G^-(\tilde{b}).v_\lambda$ and $B(-\lambda) \to T_{-\lambda} \otimes B(-\infty), b \mapsto t_{-\lambda} \otimes \tilde{b}$ with $g_b = G^+(\tilde{b}).v_\lambda$ are the embeddings of crystals. ([4])
- For $\lambda \in P$, we have $\{G(\dot{b}).v_\lambda\}_{\dot{b} \in B(\dot{U})} \setminus \{0\} = \{g_b\}_{b \in B(\lambda)}$. Moreover, the map $B(\lambda) \to B(\dot{U}), b \mapsto \dot{b}$ with $g_b = G(\dot{b}).v_\lambda$ is the strict embedding of crystals. ([7])
- For $\lambda \in P$ and $\tilde{b} \in B(\infty)$, we have

$$G(\tilde{b} \otimes t_\lambda \otimes \tilde{b}_\infty^\omega) = G^-(\tilde{b})a_\lambda, \text{ and } G(\tilde{b}_\infty \otimes t_\lambda \otimes \tilde{b}^\omega) = G^+(\tilde{b}^\omega)a_\lambda.$$

- The U-module isomorphism $V(\lambda) \to V(s_i\lambda)$ in Remark 2.8 sends the global basis to the global basis. ([7])

Proposition 2.22 ([24, Proposition 3.4.7, Corollary 3.4.8], [17, Theorem 1.2])**.** *Let $i \in I$ and*

$${}^i\pi : U^- = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} F_i^{(n)} \operatorname{Ker} e_i' \to \operatorname{Ker} e_i',$$

$$\pi^i : U^- = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \operatorname{Ker} {}_ie' F_i^{(n)} \to \operatorname{Ker} {}_ie',$$

be natural projections. Then, for $\tilde{b} \in B(\infty)$ *with* $\varepsilon_i(\tilde{b}) = 0$, *we have*

$${}^i\pi(G^-(\tilde{b})) = T''_{i,1}(\pi^i G^-(\tau_i(\tilde{b}))),$$

where $\tau_i : \{\tilde{b} \in B(\infty) \mid \varepsilon_i(\tilde{b}) = 0\} \to \{\tilde{b} \in B(\infty) \mid \varepsilon_i^*(\tilde{b}) = 0\}$ *is the bijection given by* $\tilde{b} \mapsto \tilde{f}_i^{\varphi_i^*(\tilde{b})}(\tilde{e}_i^*)^{\varepsilon_i^*(\tilde{b})}\tilde{b}$.

Definition 2.23. For $\dot{b} \in B(\dot{U})$ (resp. $b \in B(\lambda)(\lambda \in P)$), define an element $F(\dot{b}) \in \dot{U}^*$ (resp. $f_b \in V(\lambda)^*$) by $G(\dot{b}') \mapsto \delta_{\dot{b},\dot{b}'}$ for $\dot{b}' \in B(\dot{U})$ (resp. $g_{b'} \mapsto \delta_{b,b'}$ for $b' \in B(\lambda)$).

In the rest of this subsection, we summarize the properties of the lower global bases relevant to the structure constants and prove one lemma (Lemma 2.33) and one proposition (Proposition 2.35).

Notation 2.24. For any $\tilde{b} \in B(\infty), \dot{b} \in B(\dot{U}), i \in I$ and $k \in \mathbb{Z}_{\geq 0}$, we write

$$F_i^{(k)}G^-(\tilde{b}) = \sum_{\tilde{b}'\in B(\infty)} f_{\tilde{b},\tilde{b}'}^{(k),i} G^-(\tilde{b}'), \quad (e_i')^k(G^-(\tilde{b})) = \sum_{\tilde{b}'\in B(\infty)} d_{\tilde{b},\tilde{b}'}^{(k),i} G^-(\tilde{b}'),$$
$$E_i^{(k)}G(\dot{b}) = \sum_{\dot{b}'\in B(\dot{U})} E_{\dot{b},\dot{b}'}^{(k),i} G(\dot{b}'),$$

with $f_{\tilde{b},\tilde{b}'}^{(k),i}, d_{\tilde{b},\tilde{b}'}^{(k),i}, E_{\dot{b},\dot{b}'}^{(k),i} \in \mathbb{Z}[q^{\pm 1}]$.

Remark 2.25. For any $\tilde{b} \in B(\infty), \dot{b} \in B(\dot{U}), i \in I$ and $k \in \mathbb{Z}_{\geq 0}$, we have

$$G^-(\tilde{b})F_i^{(k)} = \sum_{\tilde{b}'\in B(\infty)} f_{*\tilde{b},*\tilde{b}'}^{(k),i} G^-(\tilde{b}'), \quad ({}_ie')^k(G^-(\tilde{b})) = \sum_{\tilde{b}'\in B(\infty)} d_{*\tilde{b},*\tilde{b}'}^{(k),i} G^-(\tilde{b}'),$$
$$E_i^{(k)}.F(\dot{b}) = \sum_{\dot{b}'\in B(\dot{U})} E_{*\dot{b}',*\dot{b}}^{(k),i} F(\dot{b}').$$

Remark that the summations in the equality in $\dot{U}^*$ may be infinite.

Proposition 2.26 ([6, Proposition 5.3.1], [7, Proposition 6.4.3])**.** *The following statements hold:*

(i) *For* $\tilde{b}, \tilde{b}' \in B(\infty)$, $i \in I$ *and* $k \in \mathbb{Z}_{\geq 0}$, *we have*

$$f_{\tilde{b},\tilde{b}'}^{(k),i} \begin{cases} = \begin{bmatrix} \varepsilon_i(\tilde{b}) + k \\ k \end{bmatrix}_i & \text{if } \tilde{b}' = \tilde{f}_i^k\tilde{b} \\ \in qq_i^{-k(\varepsilon_i(\tilde{b}')-k)}\mathbb{Z}[q] & \text{if } \varepsilon_i(\tilde{b}') > \varepsilon_i(\tilde{b}) + k \\ = 0 & \text{otherwise,} \end{cases}$$

(ii) *For* $\tilde{b}, \tilde{b}' \in B(\infty)$, $i \in I$ *and* $k \in \mathbb{Z}_{\geq 0}$, *we have*

$$d_{\tilde{b},\tilde{b}'}^{(k),i} \begin{cases} = q_i^{-k\varepsilon_i(\tilde{b})+\frac{1}{2}k(k+1)} & \text{if } \tilde{b}' = \tilde{e}_i^k\tilde{b} \\ \in qq_i^{-k\varepsilon_i(\tilde{b}')-\frac{1}{2}k(k-1)}\mathbb{Z}[q] & \text{if } \varepsilon_i(\tilde{b}') > \varepsilon_i(\tilde{b}) - k \\ = 0 & \text{otherwise.} \end{cases}$$

(iii) *For $\dot{b}, \dot{b}' \in B(\dot{U}), i \in I$ and $k \in \mathbb{Z}_{\geq 0}$, we have*

$$E^{(k),i}_{\dot{b},\dot{b}'} \begin{cases} = \begin{bmatrix} \varphi_i(\dot{b}) + k \\ k \end{bmatrix}_i & \text{if } \dot{b}' = \tilde{e}_i^k \dot{b} \\ \in q q_i^{-k(\varphi_i(\dot{b}')-k)} \mathbb{Z}[q] & \text{if } \varphi_i(\dot{b}') > \varphi_i(\dot{b}) + k \text{ and } \operatorname{wt}^* \dot{b}' = \operatorname{wt}^* \dot{b} \\ = 0 & \text{otherwise,} \end{cases}$$

(iv) *For $b, b' \in B(\lambda)(\lambda \in P)$, we have*

$$f_b.E_i^{(\varphi_i(b))} = f_{\tilde{f}_i^{\varphi_i(b)} b}, \quad \text{and } f_b.E_i^{(k)} = 0 \text{ if } k > \varphi_i(b),$$
$$f_b.F_i^{(\varepsilon_i(b))} = f_{\tilde{e}_i^{\varepsilon_i(b)} b}, \quad \text{and } f_b.F_i^{(k)} = 0 \text{ if } k > \varepsilon_i(b).$$

REMARK 2.27. To show Proposition 2.26 (ii) from [6, Proposition 5.3.1] and [7, Proposition 6.4.3], we should note the following equality:

$$E_i G^-(\tilde{b}).v_\lambda = \frac{q_i^{\langle \lambda + \operatorname{wt} \tilde{b}, \alpha_i^\vee \rangle + 2} \overline{(e_i')(G^-(\tilde{b}))} - q_i^{-\langle \lambda + \operatorname{wt} \tilde{b}, \alpha_i^\vee \rangle - 2} (e_i')(G^-(\tilde{b}))}{q_i - q_i^{-1}} .v_\lambda$$

for $\lambda \in P_+$ and $\tilde{b} \in B(\infty)$. The details are left to the reader.

Proposition 2.28 ([4, Theorem 7]). *The following statements hold:*

(i) *The set $\{G^-(\tilde{b})\}_{\tilde{b} \in B(\infty); \varepsilon_i(\tilde{b}) \geq k}$ is a $\mathbb{Q}(q)$-basis of $F_i^{(k)} U^-$ for any $k \in \mathbb{Z}_{\geq 0}$.*

(ii) *The set $\{G^-(\tilde{b})\}_{\tilde{b} \in B(\infty); \varepsilon_i^*(\tilde{b}) \geq k}$ is a $\mathbb{Q}(q)$-basis of $U^- F_i^{(k)}$ for any $k \in \mathbb{Z}_{\geq 0}$.*

Corollary 2.29. *For $\tilde{b}, \tilde{b}' \in B(\infty), i \in I$ and $k \in \mathbb{Z}_{\geq 0}$, we have*

$$f^{(k),i}_{\tilde{b},\tilde{b}'} = 0 \text{ unless } \varepsilon_i^*(\tilde{b}') \geq \varepsilon_i^*(\tilde{b}).$$

Proposition 2.30 ([7, Proposition 6.4.2]). *For $\lambda \in P$, $i \in I$ and $k, l \in \mathbb{Z}_{\geq 0}$, we have*

$$U E_i^{(k)} a_\lambda + U F_i^{(l)} a_\lambda = \bigoplus_{\dot{b}} \mathbb{Q}(q) G(\dot{b}).$$

Here $\dot{b}$ ranges over $\{\dot{b} \in B(\dot{U} a_\lambda) \mid \varphi_i^*(\dot{b}) \geq k \text{ or } \varepsilon_i^*(\dot{b}) \geq l\}$.

REMARK 2.31. Let $\lambda \in P$, $i \in I$, $\dot{b} \in B(\dot{U} a_\lambda)$ and $k, l \in \mathbb{Z}_{\geq 0}$. If $\langle \lambda, \alpha_i^\vee \rangle \geq l - k$, then $\varphi_i^*(\dot{b}) \geq k$ implies $\varepsilon_i^*(\dot{b}) = \varphi_i^*(\dot{b}) + \langle \lambda, \alpha_i^\vee \rangle \geq l$. Hence, the condition $\varphi_i^*(\dot{b}) \geq k$ or $\varepsilon_i^*(\dot{b}) \geq l$ is equivalent to $\varepsilon_i^*(\dot{b}) \geq l$. Similarly, when $\langle \lambda, \alpha_i^\vee \rangle \leq l - k$, the condition $\varphi_i^*(\dot{b}) \geq k$ or $\varepsilon_i^*(\dot{b}) \geq l$ is equivalent to $\varphi_i^*(\dot{b}) \geq k$.

Corollary 2.32. *For $\dot{b}, \dot{b}' \in B(\dot{U}), i \in I$ and $k \in \mathbb{Z}_{\geq 0}$, we have*

$$E^{(k),i}_{\dot{b},\dot{b}'} = 0 \text{ unless } \varepsilon_i^*(\dot{b}') \geq \varepsilon_i^*(\dot{b}) \text{ and } \varphi_i^*(\dot{b}') \geq \varphi_i^*(\dot{b}).$$

Proof. By Proposition 2.30 and Remark 2.31,

$$U E_i^{(k)} a_{-\operatorname{wt}^* \dot{b}} + U F_i^{(\varepsilon_i^*(\dot{b}))} a_{-\operatorname{wt}^* \dot{b}} = \bigoplus_{\dot{b}'' \in B(\dot{U} a_{-\operatorname{wt}^* \dot{b}}), \varepsilon_i^*(\dot{b}'') \geq \varepsilon_i^*(\dot{b})} \mathbb{Q}(q) G(\dot{b}''),$$

for sufficiently large k. Hence, the right-hand side is a left ideal of $\dot{U}$. Similarly, the vector space $\bigoplus_{\dot{b}'' \in B(\dot{U} a_{-\operatorname{wt}^* \dot{b}}), \varphi_i^*(\dot{b}'') \geq \varphi_i^*(\dot{b})} \mathbb{Q}(q) G(\dot{b}'')$ is a left ideal of $\dot{U}$. □

Lemma 2.33. *Let $\lambda \in P$ and $i \in I$ with $\langle \lambda, \alpha_i^\vee \rangle \geq 0$. Take $b \in B(\lambda)$. Assume that there exists $\tilde{b} \in B(\infty)$ such that $G^-(\tilde{b}).v_\lambda = g_b$ and $\varepsilon_i^*(\tilde{b}) = \langle \lambda, \alpha_i^\vee \rangle$. Then,*

$$G^-((\tilde{e}_i^*)^{\langle \lambda, \alpha_i^\vee \rangle}\tilde{b}).v_{s_i} = g_b \text{ and } (\tilde{e}_i^*)^{\langle \lambda, \alpha_i^\vee \rangle}\tilde{b} = \tau_i(\tilde{e}_i^{\max}(\tilde{f}_i^*)^{\varphi_i(b)}\tilde{b}),$$

where $\tilde{e}_i^{\max}(\tilde{f}_i^*)^{\varphi_i(b)}\tilde{b} := \tilde{e}_i^{\varepsilon_i((\tilde{f}_i^*)^{\varphi_i(b)}\tilde{b})}(\tilde{f}_i^*)^{\varphi_i(b)}\tilde{b}$. See Definition 2.7 *for the definition of v_{s_i}.*

Proof. The equality $G^-((\tilde{e}_i^*)^{\langle \lambda, \alpha_i^\vee \rangle}\tilde{b}).v_{s_i} = g_b$ follows from Proposition 2.26, Proposition 2.28 with Remark 2.25. The details are left to the reader.

Now, $\varepsilon_i^*((\tilde{e}_i^*)^{\langle \lambda, \alpha_i^\vee \rangle}\tilde{b}) = 0 = \varepsilon_i^*(\tau_i(\tilde{e}_i^{\max}(\tilde{f}_i^*)^{\varphi_i(b)}\tilde{b}))$. Hence, by Corollary 2.16, it remains to prove that $\mathrm{wt}(\tilde{e}_i^*)^{\langle \lambda, \alpha_i^\vee \rangle}\tilde{b} = \mathrm{wt}\,\tau_i(\tilde{e}_i^{\max}(\tilde{f}_i^*)^{\varphi_i(b)}\tilde{b})$.

Let $\lambda_+ := \sum_{j\in I} \max\{\langle \lambda, \alpha_j^\vee \rangle, 0\}\varpi_j \in P_+$. Then, $G^-(\tilde{b}).v_\lambda \neq 0$ implies $G^-(\tilde{b}).v_{\lambda_+} \neq 0$. Hence, $\varphi_i(\tilde{b}) + \langle \lambda, \alpha_i^\vee \rangle = \varphi_i(\tilde{b}) + \langle \lambda_+, \alpha_i^\vee \rangle \geq 0$. Therefore, $\varphi_i(b) = \varphi_i(\tilde{b} \otimes t_\lambda \otimes \tilde{b}_\infty^\omega) = \varphi_i(\tilde{b}) + \langle \lambda, \alpha_i^\vee \rangle$.

Hence, $\varepsilon_i^*(\tilde{b}) + \varphi_i(\tilde{b}) = \langle \lambda, \alpha_i^\vee \rangle + \varphi_i(\tilde{b}) = \varphi_i(b)$ and, by Corollary 2.15, $\varepsilon_i((\tilde{f}_i^*)^{\varphi_i(b)}\tilde{b}) = \varepsilon_i(\tilde{b})$. Therefore, we obtain

$$\begin{aligned}
\mathrm{wt}\,\tau_i(\tilde{e}_i^{\max}(\tilde{f}_i^*)^{\varphi_i(b)}\tilde{b}) &= s_i(\mathrm{wt}\,\tilde{b} + (\varepsilon_i(\tilde{b}) - \varphi_i(b))\alpha_i) \\
&= s_i(\mathrm{wt}\,\tilde{b} - (\langle \lambda, \alpha_i^\vee \rangle + \langle \mathrm{wt}\,\tilde{b}, \alpha_i^\vee \rangle)\alpha_i) \\
&= \mathrm{wt}\,\tilde{b} + \langle \lambda, \alpha_i^\vee \rangle \alpha_i = \mathrm{wt}(\tilde{e}_i^*)^{\langle \lambda, \alpha_i^\vee \rangle}\tilde{b}.
\end{aligned}$$

□

Notation 2.34. For a Laurent polynomial $P \in \mathbb{Z}[q^{\pm 1}]$ and an integer $m \in \mathbb{Z}$, the degree $< m$ part of P will be denoted by $P_{<m}$.

Proposition 2.35 (Similarity of the structure constants)**.** *Recall* Notation 2.24. *For any $\tilde{b}, \tilde{b}' \in B(\infty)$, $i \in I$ and $k \in \mathbb{Z}_{\geq 0}$, we have*

$$\left(f_{\tilde{b},\tilde{b}'}^{(k),i}\right)_{<-\Delta_i k(\varepsilon'_{(k)}-1)} = \left(q_i^{\frac{1}{2}\varepsilon'_{(k)}(\varepsilon'_{(k)}-1)} \begin{bmatrix} \varepsilon_i(\tilde{b}') \\ k \end{bmatrix}_i d_{\tilde{b},\tilde{e}_i^{\max}\tilde{b}'}^{(\varepsilon'_{(k)}),i}\right)_{<-\Delta_i k(\varepsilon'_{(k)}-1)},$$

where $\varepsilon'_{(k)} := \varepsilon_i(\tilde{b}') - k$ and $\tilde{e}_i^{\max}\tilde{b}' := \tilde{e}_i^{\varepsilon_i(\tilde{b}')}\tilde{b}'$.

Proof. By Proposition 2.26, we have

$$(e_i')^{\varepsilon_i(\tilde{b}')}(F_i^{(k)}G^-(\tilde{b})) = \sum_{\substack{\tilde{b}'' \in B(\infty) \\ \varepsilon_i(\tilde{b}'') \geq \varepsilon_i(\tilde{b})+k}} \sum_{\substack{\tilde{b}''' \in B(\infty) \\ \varepsilon_i(\tilde{b}''') \geq \varepsilon_i(\tilde{b}'')-\varepsilon_i(\tilde{b}')}} f_{\tilde{b},\tilde{b}''}^{(k),i} d_{\tilde{b}'',\tilde{b}'''}^{(\varepsilon_i(\tilde{b}')),i} G^-(\tilde{b}'''). \tag{2.2}$$

Let C be the coefficient of $G^-(\tilde{e}_i^{\max}\tilde{b}')(\neq 0)$ in (2.2). Then,

$$C = \sum_{\substack{\tilde{b}'' \in B(\infty) \\ \varepsilon_i(\tilde{b}') \geq \varepsilon_i(\tilde{b}'') \geq \varepsilon_i(\tilde{b})+k}} f_{\tilde{b},\tilde{b}''}^{(k),i} d_{\tilde{b}'',\tilde{e}_i^{\max}\tilde{b}'}^{(\varepsilon_i(\tilde{b}')),i}.$$

By Proposition 2.26, we have

$$\sum_{\substack{\tilde{b}'' \in B(\infty) \\ \varepsilon_i(\tilde{b}') > \varepsilon_i(\tilde{b}'') \geq \varepsilon_i(\tilde{b})+k}} f_{\tilde{b},\tilde{b}''}^{(k),i} d_{\tilde{b}'',\tilde{e}_i^{\max}\tilde{b}'}^{(\varepsilon_i(\tilde{b}')),i} \in q_i^{-k(\varepsilon_i(\tilde{b}')-k-1)-\frac{1}{2}\varepsilon_i(\tilde{b}')(\varepsilon_i(\tilde{b}')-1)}\mathbb{Z}[q].$$

Moreover, if $\varepsilon_i(\tilde{b}') = \varepsilon_i(\tilde{b}'')$ and $d^{(\varepsilon_i(\tilde{b}')),i}_{\tilde{b}'',\tilde{e}_i^{\max}\tilde{b}'} \neq 0$, then $\tilde{e}_i^{\varepsilon_i(\tilde{b}')}\tilde{b}'' = \tilde{e}_i^{\max}\tilde{b}'(\neq 0)$ (equivalently, $\tilde{b}'' = \tilde{b}'$). Therefore, we have

$$(C)_{<(\bullet)} = \left(q_i^{-\frac{1}{2}\varepsilon_i(\tilde{b}')(\varepsilon_i(\tilde{b}')-1)} f^{(k),i}_{\tilde{b},\tilde{b}'}\right)_{<(\bullet)},$$

where $(\bullet) = -\Delta_i\left(k(\varepsilon'_{(k)} - 1) + \varepsilon_i(\tilde{b}')(\varepsilon_i(\tilde{b}') - 1)\right)/2$.

On the other hand,

$$\begin{aligned}&(e'_i)^{\varepsilon_i(\tilde{b}')}(F_i^{(k)}G^-(\tilde{b}))\\ &= \sum_{s=0}^{k} q_i^{-2\varepsilon_i(\tilde{b}')k+(\varepsilon_i(\tilde{b}')+k)s-\frac{1}{2}s(s-1)} \begin{bmatrix} \varepsilon_i(\tilde{b}') \\ s \end{bmatrix}_i F_i^{(k-s)}(e'_i)^{\varepsilon_i(\tilde{b}')-s}G^-(\tilde{b}).\end{aligned}$$

This follows from a direct computation and this calculation result is written in the reference [4, (3.1.2)]. By Proposition 2.26, we have

$$\begin{aligned}&F_i^{(k-s)}(e'_i)^{\varepsilon_i(\tilde{b}')-s}G^-(\tilde{b})\\ &= \sum_{\substack{\tilde{b}''\in B(\infty)\\ \varepsilon_i(\tilde{b}'')\geq\varepsilon_i(\tilde{b})-\varepsilon_i(\tilde{b}')+s}} \sum_{\substack{\tilde{b}'''\in B(\infty)\\ \varepsilon_i(\tilde{b}''')\geq\varepsilon_i(\tilde{b}'')+k-s}} d^{(\varepsilon_i(\tilde{b}')-s),i}_{\tilde{b},\tilde{b}''} f^{(k-s),i}_{\tilde{b}'',\tilde{b}'''} G^-(\tilde{b}''').\end{aligned}$$

We consider the case $\tilde{b}''' = \tilde{e}_i^{\max}\tilde{b}'$. The inequality $0 \geq \varepsilon_i(\tilde{b}'') + k - s$ does not hold unless $s = k$. Hence, we have

$$C = q_i^{-\varepsilon_i(\tilde{b}')k+\frac{1}{2}k(k+1)} \begin{bmatrix} \varepsilon_i(\tilde{b}') \\ k \end{bmatrix}_i d^{(\varepsilon'_{(k)}),i}_{\tilde{b},\tilde{e}_i^{\max}\tilde{b}'}.$$

Therefore,

$$\left(q_i^{-\frac{1}{2}\varepsilon_i(\tilde{b}')(\varepsilon_i(\tilde{b}')-1)} f^{(k),i}_{\tilde{b},\tilde{b}'}\right)_{<(\bullet)} = \left(q_i^{-\varepsilon_i(\tilde{b}')k+\frac{1}{2}k(k+1)} \begin{bmatrix} \varepsilon_i(\tilde{b}') \\ k \end{bmatrix}_i d^{(\varepsilon_i(\tilde{b}')-k),i}_{\tilde{b},\tilde{e}_i^{\max}\tilde{b}'}\right)_{<(\bullet)}.$$

Hence, we obtain the equality

$$\left(f^{(k),i}_{\tilde{b},\tilde{b}'}\right)_{<-\Delta_i k(\varepsilon'_{(k)}-1)} = \left(q_i^{\frac{1}{2}\varepsilon'_{(k)}(\varepsilon'_{(k)}-1)} \begin{bmatrix} \varepsilon_i(\tilde{b}') \\ k \end{bmatrix}_i d^{(\varepsilon'_{(k)}),i}_{\tilde{b},\tilde{e}_i^{\max}\tilde{b}'}\right)_{<-\Delta_i k(\varepsilon'_{(k)}-1)}.$$

□

Definition 2.36. Let $\lambda \in P_+ \cup (-P_+)$. Then, there exists a unique nondegenerate symmetric $\mathbb{Q}(q)$-bilinear form $(\ ,\ )_\lambda$ on $V(\lambda)$ such that

$$(v_\lambda, v_\lambda)_\lambda = 1 \text{ and } (x.u, v)_\lambda = (u, \psi(x).v)_\lambda \text{ for } u, v \in V(\lambda) \text{ and } x \in U.$$

For $v \in V(\lambda)$, we set $v^* := (u \mapsto (u, v)_\lambda) \in V(\lambda)^*$.

Since $\dim V(\lambda)_\mu < \infty$ for any $\mu \in P$ and $(\ ,\ )_\lambda$ is nondegenerate, there uniquely exists a basis $\{g_b^\vee\}_{b\in B(\lambda)}$ of $V(\lambda)$ such that $(g_b^\vee)^* = f_b$ for all $b \in B(\lambda)$. See Definition 2.23 for the definition of f_b.

Note that $(v_w, v_w)_\lambda = 1$ for all extremal vectors v_w $(w \in W)$. See Definition 2.7. Hence, $v_w = g_{b_w} = g^\vee_{b_w}$ for some $b_w \in B(\lambda)$.

Denote by $v_{w\lambda}$ (resp. $f_{w\lambda}$) the element $v_w \in V(\lambda)_{w\lambda}$ (resp. $v_w^* \in V(\lambda)^*_{w\lambda}$). Remark that

$\dim V(\lambda)_{w\lambda} = 1$ and $v_w = v_{w'}$ if $w\lambda = w'\lambda$ when $\lambda \in P_+ \cup (-P_+)$.

3. Representations of the quantized coordinate algebra $A_q[\mathfrak{sl}_2]$

In this section, we investigate the actions of the matrix coefficients of extremal weight modules on the infinite dimensional irreducible modules V_i, V_i' of the quantized coordinate algebras. Proposition 3.10 is one of the main calculations in this paper.

3.1. A method of calculation.

Definition 3.1. The quantized coordinate algebra $A_q[\mathfrak{sl}_2]$ of $\mathfrak{sl}_2$ is the subalgebra of $U_q(\mathfrak{sl}_2)^*$ generated by $c_{ij} := F^{\delta_{2,j}}.c^{\varpi}_{f_\varpi,v_\varpi}.E^{\delta_{2,i}}$ ($i, j \in \{1,2\}$). Here, since $I = \{*\}$, the elements E_k, F_k, ϖ_k ($k \in I$) are simply denoted by E, F, ϖ respectively.

Moreover, c_{ij}'s satisfy the following relations and, in fact, the relations of c_{ij}'s are exhausted by them:

$$\begin{gathered} c_{i1}c_{i2} = qc_{i2}c_{i1}\ (i=1,2), \quad c_{1j}c_{2j} = qc_{2j}c_{1j}\ (j=1,2), \\ [c_{12}, c_{21}] = 0, \qquad [c_{11}, c_{22}] = (q-q^{-1})c_{12}c_{21}, \\ c_{11}c_{22} - qc_{12}c_{21} = 1. \end{gathered}$$

Let $V_* := \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathbb{Q}(q)\,|m\rangle$ be an infinite dimensional $\mathbb{Q}(q)$-vector space with a basis indexed by non-negative integers. We define an $A_q[\mathfrak{sl}_2]$-module structure on V_* by

$$\begin{aligned} c_{11}.: |m\rangle &\longmapsto \begin{cases} 0 & \text{if } m=0, \\ |m-1\rangle & \text{if } m \in \mathbb{Z}_{>0}, \end{cases} \\ c_{12}.: |m\rangle &\longmapsto -q^{m+1}\,|m\rangle \qquad \text{for } m \in \mathbb{Z}_{\geq 0}, \\ c_{21}.: |m\rangle &\longmapsto q^{m}\,|m\rangle \qquad \text{for } m \in \mathbb{Z}_{\geq 0}, \\ c_{22}.: |m\rangle &\longmapsto (1-q^{2(m+1)})\,|m+1\rangle \qquad \text{for } m \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

By its construction, it is easy to see that this is a simple $A_q[\mathfrak{sl}_2]$-module. The corresponding algebra homomorphism $A_q[\mathfrak{sl}_2] \to \mathrm{End}_{\mathbb{Q}(q)}(V_*)$ is denoted by π_*.

Definition 3.2. For $i \in I$, we denote by U_i the Hopf subalgebra of U generated by $\{E_i, F_i, K_i\}$. Let $\iota_i : U_i \hookrightarrow U$ be the natural inclusion of the Hopf algebra. Then, there exists the algebra homomorphism $\iota_i^* : U^* \to U_i^*$ given by $f \mapsto f \circ \iota_i$.

We can regard $\mathbb{Q}(q) \otimes_{\mathbb{Q}(q_i)} A_{q_i}[\mathfrak{sl}_2]$ as a subalgebra of U_i^* and denote this subalgebra by A_i. The irreducible representation π_* corresponding to A_i will be denoted by π_i and its representation space will be written as $V_i = \oplus_{m\in\mathbb{Z}_{\geq 0}}\mathbb{Q}(q)\,|m\rangle_i$.

Note that $\iota_i^*(c^V_{f,v}) \in A_i$ for any integrable U-module V, $f \in V^*$ and $v \in V$. For $\lambda \in P \cup (-P_+)$ and $w \in W$, we have

$$\iota_i^*(c^\lambda_{f_{s_iw\lambda}, v_{w\lambda}}) = \begin{cases} c_{12}^{-\langle \alpha_i^\vee, w\lambda\rangle} & \text{if } \langle \alpha_i^\vee, w\lambda\rangle \leq 0, \\ c_{21}^{\langle \alpha_i^\vee, w\lambda\rangle} & \text{if } \langle \alpha_i^\vee, w\lambda\rangle \geq 0. \end{cases}$$

Definition 3.3. Let $\lambda \in P_+ \cup (-P_+)$. Set

$$V(\lambda)^\star := \sum\nolimits_{\nu\in P} V(\lambda)_\nu^* = \bigoplus\nolimits_{b\in B(\lambda)} \mathbb{Q}(q) f_b.$$

See Definition 2.23 for the definition of f_b. Note that $\dim V(\lambda)_\nu < \infty$ for all $\nu \in P$ when

$\lambda \in P_+ \cup (-P_+)$. We regard $V(\lambda)^\star$ naturally as a right U-module. The right U-module $V(\lambda)^\star$ is an irreducible integrable highest weight right U-module with highest weight λ.

Proposition 3.4 ([18, Proposition 31.2.6]). *Let V be an integrable left U-module and $\lambda_1, \lambda_2 \in P_+$.*

$$\tilde{V} := \left\{ v \in V_{\lambda_1 - \lambda_2} \;\middle|\; \begin{array}{l} F_i^{(k)}.v = 0 \text{ for all } k > \langle \lambda_1, \alpha_i^\vee \rangle, i \in I, \text{ and} \\ E_i^{(k)}.v = 0 \text{ for all } k > \langle \lambda_2, \alpha_i^\vee \rangle, i \in I \end{array} \right\}.$$

Then the maps

$$\begin{aligned} &\mathrm{Hom}_U(V(\lambda_1) \otimes V(-\lambda_2), V) \to \tilde{V}, \varrho \mapsto \varrho(v_{\lambda_1} \otimes v_{-\lambda_2}), \\ &\mathrm{Hom}_U(V(-\lambda_2) \otimes V(\lambda_1), V) \to \tilde{V}, \varrho \mapsto \varrho(v_{-\lambda_2} \otimes v_{\lambda_1}) \end{aligned}$$

are $\mathbb{Q}(q)$-linear isomorphisms.

Proposition 3.5 ([18, Proposition 31.2.6]). *Let V' be an integrable right U-module and $\lambda_1, \lambda_2 \in P_+$.*

$$\tilde{V}' := \left\{ v \in V'_{\lambda_2 - \lambda_1} \;\middle|\; \begin{array}{l} v.F_i^{(k)} = 0 \text{ for all } k > \langle \lambda_1, \alpha_i^\vee \rangle, i \in I, \text{ and} \\ v.E_i^{(k)} = 0 \text{ for all } k > \langle \lambda_2, \alpha_i^\vee \rangle, i \in I \end{array} \right\}.$$

Then the maps

$$\begin{aligned} &\mathrm{Hom}_U(V(-\lambda_1)^\star \otimes V(\lambda_2)^\star, V') \to \tilde{V}', \varrho \mapsto \varrho(f_{-\lambda_1} \otimes f_{\lambda_2}), \\ &\mathrm{Hom}_U(V(\lambda_2)^\star \otimes V(-\lambda_1)^\star, V') \to \tilde{V}, \varrho \mapsto \varrho(f_{\lambda_2} \otimes f_{-\lambda_1}) \end{aligned}$$

are $\mathbb{Q}(q)$-linear isomorphisms.

By Proposition 2.11 and 3.4, we obtain the following corollaries. The details are left to the reader.

Corollary 3.6. *Let V be an integrable left U-module and $w \in W$. Then, for any weight vector $v \in V$, there exists a homomorphism $\varrho : V(\lambda) \otimes V(w^{-1} \operatorname{wt} v - \lambda) \to V$ of U-modules given by $\varrho(v_{w\lambda} \otimes v_{w(w^{-1} \operatorname{wt} v - \lambda)}) = v$ whenever $\langle \lambda, \alpha_i^\vee \rangle$'s are sufficiently large for all $i \in I$.*

Corollary 3.7. *Let V' be an integrable right U-module and $w \in W$. Then, for any weight vector $f \in V'$, there exists a homomorphism $\varrho' : V(\lambda)^\star \otimes V(w^{-1} \operatorname{wt} f - \lambda)^\star \to V'$ of right U-modules given by $\varrho'(f_{w\lambda} \otimes f_{w(w^{-1} \operatorname{wt} f - \lambda)}) = f$ whenever $\langle \lambda, \alpha_i^\vee \rangle$'s are sufficiently large for all $i \in I$.*

From now on, we study the representations π_i.

Lemma 3.8. *Let $i \in I$ and V an integrable U-module. If $\iota_i^*(c_{f,v}^V).|0\rangle_i \neq 0$ for weight vectors $f \in V^*, v \in V$, then*

$$\operatorname{wt} f - \operatorname{wt} v \in \mathbb{Z}\alpha_i \text{ and } \langle \operatorname{wt} f + \operatorname{wt} v, \alpha_i^\vee \rangle \leq 0.$$

Proof. The fact $\operatorname{wt} f - \operatorname{wt} v \in \mathbb{Z}\alpha_i$ clearly follows from $\iota_i^*(c_{f,v}^V) \neq 0$. We can write

$$c_{f,v}^V = \sum_{m_1,m_2,m_3,m_4 \in \mathbb{Z}_{\geq 0}} a_{m_1,m_2,m_3,m_4} c_{22}^{m_1} c_{21}^{m_2} c_{12}^{m_3} c_{11}^{m_4}$$

with $a_{m_1,m_2,m_3,m_4} \in \mathbb{Q}(q)$. Then, we have $\langle \operatorname{wt} f + \operatorname{wt} v, \alpha_i^\vee \rangle = -2m_1 + 2m_4$ for any $a_{m_1,m_2,m_3,m_4} \neq 0$. Hence, if $\langle \operatorname{wt} f + \operatorname{wt} v, \alpha_i^\vee \rangle > 0$, then $m_4 > 0$ for all $a_{m_1,m_2,m_3,m_4} \neq 0$, and $\iota_i^*(c_{f,v}^V).|0\rangle = 0$ by definition. This contradicts our assumption. □

Lemma 3.9. *Let $\lambda \in P$ and $i \in I$ with $\langle \lambda, \alpha_i^\vee \rangle \geq 0$. Take $b_1, b_2 \in B(\lambda)$ with $g_{b_2} \in U^-.v_\lambda$. Let $\tilde{b}_2 \in B(\infty)$ be the element satisfying $G^-(\tilde{b}_2).v_\lambda = g_{b_2}$.*

Suppose that $\iota_i^*(c^\lambda_{f_{b_1}, g_{b_2}}) \neq 0$. Then, we have

(i) $g_{b_1} \in U^-.v_\lambda$ *and* $\mathrm{wt}\, b_1 - \mathrm{wt}\, b_2 \in \mathbb{Z}\alpha_i$,

(ii) $\frac{1}{2}\langle \mathrm{wt}\, b_1 - \mathrm{wt}\, b_2, \alpha_i^\vee \rangle \leq \varphi_i(b_1) \leq \frac{1}{2}\langle \mathrm{wt}\, b_1 + \mathrm{wt}\, b_2, \alpha_i^\vee \rangle - \langle \mathrm{wt}\, \tilde{b}_2, \varpi_i^\vee \rangle$.

Proof. The statement (i) follows from Lemma 3.8 and the fact that

($*$) The subspace $U^-.v_\lambda$ is spanned by the global basis elements and stable under the U_i-action when $\langle \lambda, \alpha_i^\vee \rangle \geq 0$. (Note that $E_i.v_\lambda = 0$.)

Now, $\langle f_{b_1}, E_i^{(k)} F_i^{(l)}.g_{b_2} \rangle \neq 0$ for some $k, l \in \mathbb{Z}_{\geq 0}$ with $k - l = \frac{1}{2}\langle \mathrm{wt}\, b_1 - \mathrm{wt}\, b_2, \alpha_i^\vee \rangle$. Hence, $f_{b_1}.E_i^{(k)} \neq 0$. By Proposition 2.26, we have $\varphi_i(b_1) \geq k \geq \frac{1}{2}\langle \mathrm{wt}\, b_1 - \mathrm{wt}\, b_2, \alpha_i^\vee \rangle$.

It remains to prove that $\varphi_i(b_1) \leq \frac{1}{2}\langle \mathrm{wt}\, b_1 + \mathrm{wt}\, b_2, \alpha_i^\vee \rangle - \langle \mathrm{wt}\, \tilde{b}_2, \varpi_i^\vee \rangle$. By the statement (i), the fact ($*$) and Proposition 2.26, we have $g_{\tilde{e}_i^{\varepsilon_i(b_1)} b_1} \in U^-.v_\lambda$. Hence, $-Q_+ \ni \mathrm{wt}\, b_1 + \varepsilon_i(b_1)\alpha_i - \lambda = \mathrm{wt}\, b_1 - \mathrm{wt}\, b_2 + \mathrm{wt}\, \tilde{b}_2 + (\varphi_i(b_1) - \langle \mathrm{wt}\, b_1, \alpha_i^\vee \rangle)\alpha_i$. Note that $\mathrm{wt}\, b_1 - \mathrm{wt}\, b_2 \in \mathbb{Z}\alpha_i$. Focusing on the coefficient of α_i, we obtain the desired inequality. □

The following proposition is one of the most important assertions in this paper, but it looks complicated. Hence, we indicate here the places where each of assertions will be used:

- the former statement, explicit description of $p^{(k),i}_{b_1,b_2}$, will be used only in the proof of Corollary 3.12,
- the latter statement, estimation of pole orders of the Laurent polynomials $p^{(k),i}_{b_1,b_2}$, will be used only in the proof of Corollary 3.11 and 3.12.

In fact, we will not use Proposition 3.10 itself but use Corollary 3.11 and 3.12 in the subsequent sections.

Proposition 3.10 (A method of calculation)**.** *Let $\lambda \in P$ and $i \in I$ with $\langle \lambda, \alpha_i^\vee \rangle \geq 0$. Take $b_1, b_2 \in B(\lambda)$ with $\mathrm{wt}\, b_1 - \mathrm{wt}\, b_2 = -n\alpha_i$ for some $n \in \mathbb{Z}$ such that $\varphi_i(b_1) \geq -n$. Assume that there exists $\tilde{b}_l \in B(\infty)$ such that $G^-(\tilde{b}_l).v_\lambda = g_{b_l} (l = 1, 2)$. Note that these assumptions are necessary for the condition that $\iota_i^*(c^\lambda_{f_{b_1}, g_{b_2}}) \neq 0$ for $g_{b_2} \in U^-.v_\lambda$. See* Lemma 3.9.

Set

$$\lambda_1 = \sum_{j \in I} \varepsilon_j \varpi_j \in P_+ \text{ with } \varepsilon_j \begin{cases} = \varepsilon_i(b_1) & \text{if } j = i \\ \geq \varepsilon_j(b_1) & \text{if } j \in I \setminus \{i\}. \end{cases}$$

Then we have

$$\iota_i^*(c^\lambda_{f_{b_1}, g_{b_2}}).|0\rangle_i = \sum_{k=0}^{\varphi_i(b_1)} \left((-q_i)^{\varphi_k} q_i^{l_k} \sum_{\tilde{b}', \tilde{b}'' \in B(\infty)} L_{k, \tilde{b}', \tilde{b}''} \prod_{l=1}^{m_{b_1,b_2}} (1 - q_i^{2l}) \right) |m_{b_1,b_2}\rangle_i$$

$$(=: \sum_{k=0}^{\varphi_i(b_1)} p^{(k),i}_{b_1,b_2} |m_{b_1,b_2}\rangle_i \text{ for short,})$$

where

$$\varphi_k := \varphi_i(b_1) - k \text{ and } m_{b_1,b_2} := -\langle \mathrm{wt}\, b_1 + \mathrm{wt}\, b_2, \alpha_i^\vee \rangle/2,$$
$$l_k := m_{b_1,b_2}(\varphi_i(b_1) + n) + \varphi_k\langle \lambda, \alpha_i^\vee \rangle - \varphi_k(\varphi_k + 1)/2,$$
$$L_{k,\tilde{b}',\tilde{b}''} := d^{(\varphi_k),i}_{*\tilde{b}_2,*\tilde{b}'} f^{(\varphi_i(b_1)+n),i}_{\tilde{b}',\tilde{b}''} E^{(k),i}_{*(\tilde{b}'')\otimes t_{\lambda_1}\otimes \tilde{b}^{\omega}_{\infty}, *(\tilde{b}_1)\otimes t_{\lambda_1}\otimes \tilde{b}^{\omega}_{\infty}}.$$

Recall Notation 2.24. *Moreover*

$$p^{(k),i}_{b_1,b_2} \in q_i^{\Pi^{(k),i}_{b_1,b_2}} \mathbb{Z}[q],$$

where

$$\Pi^{(k),i}_{b_1,b_2} := k^2 + (\varepsilon_{b_1,b_2} - \varepsilon_i^*(\tilde{b}_1) - 1)k + \varphi_i(b_1)(\langle \lambda, \alpha_i^\vee \rangle - \varphi_i(b_1) - \varepsilon_{b_1,b_2} + 1),$$
$$\varepsilon_{b_1,b_2} := \min\{\varepsilon_i^*(\tilde{b}_1), -\langle \mathrm{wt}\, \tilde{b}_2, \varpi_i^\vee \rangle - \varphi_i(b_1)\}.$$

Before proving Proposition 3.10, we show its two corollaries.

Corollary 3.11. *We follow the notation and the assumptions in* Proposition 3.10. *Then*

$$p^{(k),i}_{b_1,b_2} \in q_i^{-\varphi_i(b_1)^2} \mathbb{Z}[q]$$

for $k = 0, \ldots, \varphi_i(b_1)$.

Proof. It suffices to show that $\Pi^{(k),i}_{b_1,b_2} \geq -\varphi_i(b_1)^2$. Here we regard $\Pi^{(k),i}_{b_1,b_2}$ as the quadratic function of k. Now $\varepsilon_{b_1,b_2} - \varepsilon_i^*(\tilde{b}_1) - 1 \leq \varepsilon_i^*(\tilde{b}_1) - \varepsilon_i^*(\tilde{b}_1) - 1 = -1$ and $\Pi^{(\varphi_i(b_1)),i}_{b_1,b_2} = \varphi_i(b_1)(\langle \lambda, \alpha_i^\vee \rangle - \varepsilon_i^*(\tilde{b}_1)) \geq 0$ by Proposition 2.28 (ii). Therefore we have $\Pi^{(k),i}_{b_1,b_2} \geq -\varphi_i(b_1)^2$ for $k = 0, \ldots, \varphi_i(b_1)$ by the shapes of graphs of quadratic functions with leading term k^2. □

Corollary 3.12. *We follow the notation and the assumptions in* Proposition 3.10. *Furthermore, we assume that* $\lambda_i := \langle \lambda, \alpha_i^\vee \rangle \geq -3\langle \mathrm{wt}\, \tilde{b}_2, \varpi_i^\vee \rangle$. *Then,*

$$(p^{(k),i}_{b_1,b_2})_{<\Delta_i(\lambda_i+3\langle \mathrm{wt}\, \tilde{b}_2, \varpi_i^\vee \rangle)} = 0 \text{ unless } k = \varphi_i(b_1) \text{ and } \varepsilon_i^*(\tilde{b}_1) = \lambda_i.$$

Recall Notations 2.1 *and* 2.34. *Moreover, if* $k = \varphi_i(b_1)$ *and* $\varepsilon_i^*(\tilde{b}_1) = \lambda_i$, *then*

$$(p^{(\varphi_i(b_1)),i}_{b_1,b_2})_{<\Delta_i(\lambda_i+2\langle \mathrm{wt}\, \tilde{b}_2, \varpi_i^\vee \rangle)} = \left(q_i^{\frac{1}{2}m_{b_1,b_2}(m_{b_1,b_2}-1)} d^{(m_{b_1,b_2}),i}_{\tilde{b}_2, \tau_i^{-1}(\tilde{b}_1^{(i)})} \right)_{<\Delta_i(\lambda_i+2\langle \mathrm{wt}\, \tilde{b}_2, \varpi_i^\vee \rangle)}.$$

where $\tilde{b}_1^{(i)} := (\tilde{e}_i^*)^{\varepsilon_i^*(\tilde{b}_1)}\tilde{b}_1 = (\tilde{e}_i^*)^{\lambda_i}\tilde{b}_1$. *See* Proposition 2.22 *for the definition of* τ_i.

REMARK 3.13. By Proposition 2.26, we have

$$q_i^{\frac{1}{2}m_{b_1,b_2}(m_{b_1,b_2}-1)} d^{(m_{b_1,b_2}),i}_{\tilde{b}_2, \tau_i^{-1}(\tilde{b}_1^{(i)})} \in \mathbb{Z}[q].$$

Proof of Corollary 3.12. By Corollary 2.15 and the assumption $\lambda_i \geq -3\langle \mathrm{wt}\, \tilde{b}_2, \varpi_i^\vee \rangle$, we have

$$\varepsilon_i^*(\tilde{b}_1) \geq -\varphi_i(\tilde{b}_1) = \lambda_i - \varphi_i(b_1) \geq -\langle \mathrm{wt}\, \tilde{b}_2, \varpi_i^\vee \rangle - \varphi_i(b_1).$$

Hence, $\varepsilon_{b_1,b_2} = -\langle \mathrm{wt}\, \tilde{b}_2, \varpi_i^\vee \rangle - \varphi_i(b_1) = -\langle \mathrm{wt}\, \tilde{b}_2, \varpi_i^\vee \rangle - \lambda_i - \varphi_i(\tilde{b}_1)$.

We first consider the case $k \neq \varphi_i(b_1)$. We may assume $m_{b_1,b_2} \geq 0$ since $p^{(k),i}_{b_1,b_2} = 0$ otherwise by construction. Then,

$$
\begin{aligned}
-\frac{1}{2}(\varepsilon_{b_1,b_2}-\varepsilon_i^*(\tilde{b}_1)-1) &= \frac{1}{2}(\langle \mathrm{wt}\,\tilde{b}_2, \varpi_i^\vee\rangle + \lambda_i + \varphi_i(\tilde{b}_1) + \varepsilon_i^*(\tilde{b}_1)+1)\\
&\geq \frac{1}{2}(\langle \mathrm{wt}\,\tilde{b}_2, \varpi_i^\vee\rangle + \lambda_i + 1)\\
&\geq -\langle \mathrm{wt}\,\tilde{b}_2, \varpi_i^\vee\rangle + \frac{1}{2} > \varphi_i(b_1).
\end{aligned}
$$

The first inequality follows from Corollary 2.15, and the last inequality follows from Lemma 3.9 (ii). (Note that we do not assume $\iota_i^*(c^\lambda_{f_{b_1},g_{b_2}}).|0\rangle_i \neq 0$ now, but the inequalities in Lemma 3.9 (ii) follows from the assumptions in Proposition 3.10.) Therefore, $\Pi^{(0),i}_{b_1,b_2} > \Pi^{(1),i}_{b_1,b_2} > \cdots > \Pi^{(\varphi_i(b_1)-1),i}_{b_1,b_2}$. Hence, $(p^{(k),i}_{b_1,b_2})_{<\Delta_i(\Pi^{(\varphi_i(b_1)-1),i}_{b_1,b_2}-2)} = 0$.

Moreover, by $\varepsilon_i^*(\tilde{b}_1) \geq \lambda_i - \varphi_i(b_1)$ and $-\varphi_i(b_1) \geq \langle \mathrm{wt}\,\tilde{b}_2, \varpi_i^\vee\rangle$ (Lemma 3.9 (ii)), we have

$$
\begin{aligned}
\Pi^{(\varphi_i(b_1)-1),i}_{b_1,b_2} - 2 &= -\varphi_i(b_1) + \varepsilon_i^*(\tilde{b}_1) + \langle \mathrm{wt}\,\tilde{b}_2, \varpi_i^\vee\rangle + \varphi_i(b_1)(\lambda_i - \varepsilon_i^*(\tilde{b}_1))\\
&\geq \lambda_i + 3\langle \mathrm{wt}\,\tilde{b}_2, \varpi_i^\vee\rangle.
\end{aligned}
$$

Note that $\lambda_i - \varepsilon_i^*(\tilde{b}_1) \geq 0$ by Proposition 2.28. Hence, $(p^{(k),i}_{b_1,b_2})_{<\Delta_i(\lambda_i+3\langle \mathrm{wt}\,\tilde{b}_2,\varpi_i^\vee\rangle)} = 0$.

Next, we consider the case $k = \varphi_i(b_1)$ ($\varphi_k = 0$). Then, by Proposition 3.10,

$$
p^{(\varphi_i(b_1)),i}_{b_1,b_2} = q_i^{m_{b_1,b_2}(\varphi_i(b_1)+n)} \sum_{\tilde{b}'\in B(\infty)} f^{(\varphi_i(b_1)+n),i}_{\tilde{b}_2,\tilde{b}'} E^{(\varphi_i(b_1)),i}_{*(\tilde{b}')\otimes t_{\lambda_1}\otimes \tilde{b}^\omega_\infty, *(\tilde{b}_1)\otimes t_{\lambda_1}\otimes\tilde{b}^\omega_\infty} \prod_{l=1}^{m_{b_1,b_2}}(1-q_i^{2l}).
$$

By the way, $\varphi_i(*(\tilde{b}_1)\otimes t_{\lambda_1}\otimes \tilde{b}^\omega_\infty) = \varphi_i^*(\tilde{b}_1) + \varepsilon_i(\tilde{b}_1) = \varphi_i(\tilde{b}_1) + \varepsilon_i^*(\tilde{b}_1) = \varphi_i(b_1) - (\lambda_i - \varepsilon_i^*(\tilde{b}_1)) \leq \varphi_i(b_1)$. The second equality follows from Corollary 2.15. Hence, by Proposition 2.26,

$$
E^{(\varphi_i(b_1)),i}_{*(\tilde{b}')\otimes t_{\lambda_1}\otimes \tilde{b}^\omega_\infty, *(\tilde{b}_1)\otimes t_{\lambda_1}\otimes\tilde{b}^\omega_\infty} = \begin{cases} 1 & \text{if } \varepsilon_i^*(\tilde{b}_1) = \lambda_i \text{ and } \tilde{b}' = (\tilde{f}_i^*)^{\varphi_i(b_1)}\tilde{b}_1 \\ 0 & \text{otherwise.}\end{cases}
$$

This proves the first half of the statement. From now on, we further assume that $\varepsilon_i^*(\tilde{b}_1) = \lambda_i$. Set

$$
\begin{aligned}
N &:= -(\varphi_i(b_1)+n)(\varepsilon_i((\tilde{f}_i^*)^{\varphi_i(b_1)}\tilde{b}_1) - \varphi_i(b_1) - n - 1)\\
&= -(\varphi_i(b_1)+n)(\varepsilon_i(\tilde{b}_1) - \varphi_i(b_1) - n - 1) \text{ by Corollary 2.15,}\\
&= -(\varphi_i(b_1)+n)(m_{b_1,b_2}-1)\\
&= -(\varphi_i(b_1)+n)m_{b_1,b_2} + \varphi_i(b_1) + m_{b_1,b_2} + \langle \mathrm{wt}\, b_2, \alpha_i^\vee\rangle
\end{aligned}
$$

cf. Proposition 2.35. Note that $\varepsilon_i(\tilde{b}_1) - \varphi_i(b_1) - n = m_{b_1,b_2}$ and we may assume that $m_{b_1,b_2} \geq 0$. Hence,

$$
\begin{aligned}
N &\geq -(\varphi_i(b_1)+n)m_{b_1,b_2} + \langle \mathrm{wt}\, b_2, \alpha_i^\vee\rangle\\
&\geq -(\varphi_i(b_1)+n)m_{b_1,b_2} + \lambda_i + 2\langle \mathrm{wt}\,\tilde{b}_2, \varpi_i^\vee\rangle.
\end{aligned}
$$

Then, by Proposition 2.26 (ii), Lemma 2.33 and Proposition 2.35,

$$
\begin{aligned}
&(p^{(\varphi_i(b_1)),i}_{b_1,b_2})_\blacklozenge\\
&= \left(q_i^{m_{b_1,b_2}(\varphi_i(b_1)+n)} f^{(\varphi_i(b_1)+n),i}_{\tilde{b}_2,(\tilde{f}_i^*)^{\varphi_i(b_1)}\tilde{b}_1} \prod_{l=1}^{m_{b_1,b_2}}(1-q_i^{2l})\right)_\blacklozenge
\end{aligned}
$$

$$
\begin{aligned}
&= \left(q_i^{m_{b_1,b_2}(\varphi_i(b_1)+n)} (f^{(\varphi_i(b_1)+n),i}_{\tilde{b}_2,(\tilde{f}_i^*)^{\varphi_i(b_1)}\tilde{b}_1})_{<\Delta_i N} \prod_{l=1}^{m_{b_1,b_2}} (1-q_i^{2l}) \right)_{\blacklozenge} \\
&= \left(q_i^{m_{b_1,b_2}(\varphi_i(b_1)+n)} \left(q_i^{\frac{1}{2}m_{b_1,b_2}(m_{b_1,b_2}-1)} \begin{bmatrix} \varepsilon_i(b_1) \\ m_{b_1,b_2} \end{bmatrix}_i d^{(m_{b_1,b_2}),i}_{\tilde{b}_2,\tau_i^{-1}(\tilde{b}_1^{(i)})} \right)_{<\Delta_i N} \prod_{l=1}^{m_{b_1,b_2}} (1-q_i^{2l}) \right)_{\blacklozenge} \\
&= \left(\left(\prod_{l=1}^{m_{b_1,b_2}} (1-q_i^{2(\varphi_i(b_1)+n+l)}) q_i^{\frac{1}{2}m_{b_1,b_2}(m_{b_1,b_2}-1)} d^{(m_{b_1,b_2}),i}_{\tilde{b}_2,\tau_i^{-1}(\tilde{b}_1^{(i)})} \right)_{<\Delta_i(\varphi_i(b_1)+n)} \right)_{\blacklozenge} \\
&= \left(q_i^{\frac{1}{2}m_{b_1,b_2}(m_{b_1,b_2}-1)} d^{(m_{b_1,b_2}),i}_{\tilde{b}_2,\tau_i^{-1}(\tilde{b}_1^{(i)})} \right)_{\blacklozenge},
\end{aligned}
$$

where $()_{\blacklozenge}$ denotes $(\ )_{<\Delta_i(\lambda_i+2\langle \mathrm{wt}\,\tilde{b}_2,\varpi_i^\vee\rangle)}$. This proves the corollary. □

The rest of this subsection is devoted to a proof of Proposition 3.10. Proof of Proposition 3.10. Set $\lambda_2 := \lambda_1 + \mathrm{wt}\, b_1$. Then, $\langle \lambda_2, \alpha_j^\vee\rangle = \varepsilon_j + \langle \mathrm{wt}\, b_1, \alpha_j^\vee\rangle \geq \varepsilon_j(b_1) + \langle \mathrm{wt}\, b_1, \alpha_j^\vee\rangle = \varphi_j(b_1) \geq 0$ for all $j \in I$. Hence, $\lambda_2 \in P_+$.

Therefore, $f_{b_1}.F_j^{(k)} = 0$ for all $k > \langle \lambda_1, \alpha_j^\vee\rangle$ and $f_{b_1}.E_j^{(k)} = 0$ for all $k > \langle \lambda_2, \alpha_j^\vee\rangle$ by Proposition 2.26 for all $j \in I$. Hence, by Proposition 3.5, there exists a homomorphism $\Psi : V(-\lambda_1)^\star \otimes V(\lambda_2)^\star \to V(\lambda)^*$ of right U-modules given by $f_{-\lambda_1} \otimes f_{\lambda_2} \mapsto f_{b_1}$. (Note that $V(\lambda)^*$ might not be integrable but the right U-submodule generated by f_{b_1} in $V(\lambda)^*$ is integrable.) Then, the dual $\mathbb{Q}(q)$-linear map $\Psi^* : (V(\lambda)^*)^* \to (V(-\lambda_1)^\star \otimes V(\lambda_2)^\star)^*$ given by $f \mapsto f \circ \Psi$ is a homomorphism of left U-modules.

For $\lambda' \in P$ and $b \in B(\lambda')$, define $\mathrm{ev}_b \in (V(\lambda')^*)^*$ by $f \mapsto \langle f, g_b\rangle$. Then, we have

$$
\Psi^*(\mathrm{ev}_{b_2}) = \sum_{b' \in B(-\lambda_1), b'' \in B(\lambda_2)} c_{b',b''}\, \mathrm{ev}_{b'} \otimes \mathrm{ev}_{b''} \ \text{ with } c_{b',b''} \in \mathbb{Q}(q).
$$

Note that this is possibly infinite sum but well-defined in $(V(-\lambda_1)^\star \otimes V(\lambda_2)^\star)^*$.

Then, we have

$$
c^\lambda_{f_{b_1},g_{b_2}} = \sum_{b' \in B(-\lambda_1), b'' \in B(\lambda_2)} c_{b',b''} c^{-\lambda_1}_{f_{-\lambda_1},g_{b'}} c^{\lambda_2}_{f_{\lambda_2},g_{b''}}.
$$

This is possibly infinite sum but well-defined in U^*. Moreover, by Lemma 3.8,

$$
\iota_i^*(c^\lambda_{f_{b_1},g_{b_2}}).|0\rangle_i = \sum_{b' \in B(-\lambda_1)} c_{b',v_{s_i\lambda_2}} \iota_i^*(c^{-\lambda_1}_{f_{-\lambda_1},g_{b'}}) \iota_i^*(c^{\lambda_2}_{f_{\lambda_2},v_{s_i\lambda_2}}).|0\rangle_i.
$$

Hence we investigate the element $\sum_{b' \in B(-\lambda_1)} c_{b',v_{s_i\lambda_2}} \iota_i^*(c^{-\lambda_1}_{f_{-\lambda_1},g_{b'}})$. We now have $\Psi^*(\mathrm{ev}_{b_2}) = \Psi^*(G^-(\tilde{b}_2).\, \mathrm{ev}_{v_\lambda}) = G^-(\tilde{b}_2).\Psi^*(\mathrm{ev}_{v_\lambda})$.

Set

$$
\begin{aligned}
\Psi^*(\mathrm{ev}_{v_\lambda}) &= \sum_{b' \in B(-\lambda_1), b'' \in B(\lambda_2)} c'_{b',b''}\, \mathrm{ev}_{b'} \otimes \mathrm{ev}_{b''} \ \text{ with } c'_{b',b''} \in \mathbb{Q}(q), \text{ and} \\
g'_k &:= \sum_{b' \in B(-\lambda_1)} c'_{b',F_i^{(k)}.v_{\lambda_2}} g_{b'} \in V(-\lambda_1).
\end{aligned}
$$

Note that if $c'_{b',F_i^{(k)}.v_{\lambda_2}} \neq 0$ then $\mathrm{wt}\, b' = \lambda - \lambda_2 + k\alpha_i$, and $B(-\lambda_1)_{\lambda-\lambda_2+k\alpha_i}$ is a finite set. Since $\langle \lambda, \alpha_i^\vee\rangle \geq 0$, we have $E_i.\Psi^*(\mathrm{ev}_{v_\lambda}) = \Psi^*(E_i.\, \mathrm{ev}_{v_\lambda}) = 0$. Thus, $E_i.(\sum_{k=0}^{\langle \lambda_2,\alpha_i^\vee\rangle} g'_k \otimes F_i^{(k)}.v_{\lambda_2}) = 0$. Therefore,

$$g'_k = (-1)^k q_i^{-k\varphi_i(b_1)+k(k-1)} \begin{bmatrix} \varphi_i(b_1) \\ k \end{bmatrix}_i^{-1} E_i^{(k)}.g'_0,$$

for $k = 0, \dots, \langle \lambda_2, \alpha_i^\vee \rangle = \varphi_i(b_1)$, which are easily checked by induction on k.

Write

$$\Delta(G^-(\tilde{b}_2)) = \sum_{l=0}^{\infty} q_i^{\frac{1}{2}l(l-1)} G_l K_{l\alpha_i} \otimes F_i^{(l)} + \sum_{\substack{G_{(1)},G_{(2)}:\text{homogeneous} \\ \text{wt}\, G_{(2)} \notin \mathbb{Z}_{\leq 0}\alpha_i}} G_{(1)} K_{-\text{wt}\, G_{(2)}} \otimes G_{(2)},$$

here $G_l := ({}_ie')^l(G^-(\tilde{b}_2))$. See Definition 2.5. Then,

$$\begin{aligned} \sum_{b' \in B(-\lambda_1)} c_{b',v_{s_i\lambda_2}} g_{b'} &= \sum_{k=0}^{\varphi_i(b_1)} q_i^{\frac{1}{2}\varphi_k(\varphi_k-1)} \begin{bmatrix} \varphi_i(b_1) \\ k \end{bmatrix}_i G_{\varphi_k} K_{\varphi_k\alpha_i}.g'_k \\ &= \sum_{k=0}^{\varphi_i(b_1)} (-q_i)^{-k} q_i^{\varphi_k\langle\lambda,\alpha_i^\vee\rangle - \frac{1}{2}\varphi_k(\varphi_k+1)} G_{\varphi_k} E_i^{(k)}.g'_0, \end{aligned}$$

where $\varphi_k := \varphi_i(b_1) - k$. Set $h_k := G_{\varphi_k} E_i^{(k)}.g'_0$ $(k = 0, \dots, \varphi_i(b_1))$. Since $h_k \in V(-\lambda_1)$ and $\text{wt}\, h_k = -\lambda_1 + (\varphi_i(b_1) + n)\alpha_i$, we have $h_k = c_k E_i^{(\varphi_i(b_1)+n)}.v_{-\lambda_1}$ for some $c_k \in \mathbb{Q}(q)$.

Claim 1.

$$\begin{aligned} \iota_i^*(c^{-\lambda_1}_{f_{-\lambda_1},h_k}) &= c_k \begin{bmatrix} \varepsilon_i(b_1) \\ \varphi_i(b_1) + n \end{bmatrix}_i c_{21}^{\varphi_i(b_1)+n} c_{22}^{m_{b_1,b_2}} \\ &= \langle f_{-\lambda_1}, F_i^{(\varphi_i(b_1)+n)}.h_k \rangle c_{21}^{\varphi_i(b_1)+n} c_{22}^{m_{b_1,b_2}}. \end{aligned}$$

Proof of Claim 1. We have

$$\iota_i^*(c^{-\lambda_1}_{f_{-\lambda_1},h_k}) = c_k E_i^{(\varphi_i(b_1)+n)}.\iota_i^*(c^{-\lambda_1}_{f_{-\lambda_1},v_{-\lambda_1}}) = c_k E_i^{(\varphi_i(b_1)+n)}.c_{22}^{\varepsilon_i(b_1)}.$$

The direct computation of the right-hand side now completes the proof. Note that $\varepsilon_i(b_1) - \varphi_i(b_1) - n = m_{b_1,b_2}$. □

We prepare one more claim on the element $c^{-\lambda_1}_{f_{-\lambda_1},g'_0}$.

Claim 2. *Set* $\dot{b}_1 := \tilde{b}_1 \otimes t_{-\lambda_1 - \text{wt}\, \tilde{b}_1} \otimes \tilde{b}^\omega_\infty = *(*(\tilde{b}_1) \otimes t_{\lambda_1} \otimes \tilde{b}^\omega_\infty) \in B(\infty) \otimes T_{\lambda-\lambda_2} \otimes B(-\infty)$. *See* Proposition 2.17. *Then,* $c^{-\lambda_1}_{f_{-\lambda_1},g'_0} = F(\dot{b}_1)$ *as an element of* $\dot{U}^*$.

Proof of Claim 2. Let $\tilde{b} \otimes t_\mu \otimes \tilde{b}' \in B(\infty) \otimes T_\mu \otimes B(-\infty) (\mu \in P)$. By Proposition 2.17 and Proposition 2.30, we have $f_{-\lambda_1}.G(\tilde{b} \otimes t_\mu \otimes \tilde{b}') = f_{-\lambda_1}.* (G(*(\tilde{b}) \otimes t_{-\mu-\text{wt}\,\tilde{b}-\text{wt}\,\tilde{b}'} \otimes *(\tilde{b}'))) = 0$ unless $\tilde{b}' = \tilde{b}^\omega_\infty$ and $\mu = -\lambda_1 - \text{wt}\, \tilde{b}$. Hence,

$$\langle f_{-\lambda_1}, G(\tilde{b} \otimes t_\mu \otimes \tilde{b}').g'_0 \rangle = \langle (f_{-\lambda_1}.G(\tilde{b} \otimes t_\mu \otimes \tilde{b}')), g'_0 \rangle = 0$$

unless $\tilde{b}' = \tilde{b}^\omega_\infty$ and $\mu = -\lambda_1 - \text{wt}\, \tilde{b}$. Moreover, for $\tilde{b} \in B(\infty)$, we have

$$\begin{aligned} \langle f_{-\lambda_1}, G(\tilde{b} \otimes t_{-\lambda_1-\text{wt}\,\tilde{b}} \otimes \tilde{b}^\omega_\infty).g'_0 \rangle &= \delta_{-\lambda_1-\text{wt}\,\tilde{b}, -\lambda_1-\text{wt}\,\tilde{b}_1} \langle f_{-\lambda_1}, G^-(\tilde{b}).g'_0 \rangle \\ &= \delta_{-\lambda_1-\text{wt}\,\tilde{b}, -\lambda_1-\text{wt}\,\tilde{b}_1} \langle f_{-\lambda_1} \otimes f_{\lambda_2}, G^-(\tilde{b}).(g'_0 \otimes v_{\lambda_2}) \rangle \\ &= \delta_{-\lambda_1-\text{wt}\,\tilde{b}, -\lambda_1-\text{wt}\,\tilde{b}_1} \langle f_{-\lambda_1} \otimes f_{\lambda_2}, G^-(\tilde{b}).\Psi^*(\text{ev}_\lambda) \rangle \end{aligned}$$

$$= \delta_{-\lambda_1 - \mathrm{wt}\,\tilde{b}, -\lambda_1 - \mathrm{wt}\,\tilde{b}_1} \langle f_{b_1}, G^-(\tilde{b}).v_\lambda \rangle$$
$$= \delta_{-\lambda_1 - \mathrm{wt}\,\tilde{b}, -\lambda_1 - \mathrm{wt}\,\tilde{b}_1} \delta_{\tilde{b},\tilde{b}_1}.$$

□

Combining all the above results, we obtain

$$\begin{aligned}
&\iota_i^*(c^\lambda_{f_{b_1}, g_{b_2}}).|0\rangle_i \\
&= \sum_{b' \in B(-\lambda_1)} c_{b', v_{s_i \lambda_2}} \iota_i^*(c^{-\lambda_1}_{f_{-\lambda_1}, g_{b'}}) \iota_i^*(c^{\lambda_2}_{f_{\lambda_2}, v_{s_i\lambda_2}}).|0\rangle_i \\
&= \sum_{k=0}^{\varphi_i(b_1)} (-q_i)^{-k} q_i^{\varphi_k \langle \lambda, \alpha_i^\vee \rangle - \frac{1}{2}\varphi_k(\varphi_k+1)} \iota_i^*(c^{-\lambda_1}_{f_{-\lambda_1}, h_k}) \iota_i^*(c^{\lambda_2}_{f_{\lambda_2}, v_{s_i\lambda_2}}).|0\rangle_i \\
&= \sum_{k=0}^{\varphi_i(b_1)} (-q_i)^{-k} q_i^{\varphi_k \langle \lambda, \alpha_i^\vee \rangle - \frac{1}{2}\varphi_k(\varphi_k+1)} \langle f_{-\lambda_1}, F_i^{(\varphi_i(b_1)+n)}.h_k \rangle c_{21}^{\varphi_i(b_1)+n} c_{22}^{m_{b_1,b_2}} c_{12}^{\varphi_i(b_1)}.|0\rangle_i \\
&= \sum_{k=0}^{\varphi_i(b_1)} (-q_i)^{\varphi_k} q_i^{l_k} \langle f_{-\lambda_1}, F_i^{(\varphi_i(b_1)+n)} G_{\varphi_k} E_i^{(k)}.g_0' \rangle \prod_{l=1}^{m_{b_1,b_2}} (1 - q_i^{2l}) |m_{b_1,b_2}\rangle_i \\
&= \sum_{k=0}^{\varphi_i(b_1)} (-q_i)^{\varphi_k} q_i^{l_k} \langle F(\dot{b}_1), F_i^{(\varphi_i(b_1)+n)} G_{\varphi_k} a_{\lambda - \lambda_2 + k\alpha_i} E_i^{(k)} \rangle \prod_{l=1}^{m_{b_1,b_2}} (1 - q_i^{2l}) |m_{b_1,b_2}\rangle_i \\
&= \sum_{k=0}^{\varphi_i(b_1)} (-q_i)^{\varphi_k} q_i^{l_k} \sum_{\tilde{b}', \tilde{b}'' \in B(\infty)} L_{k, \tilde{b}', \tilde{b}''} \prod_{l=1}^{m_{b_1,b_2}} (1 - q_i^{2l}) |m_{b_1,b_2}\rangle_i,
\end{aligned}$$

where

$$\begin{aligned}
l_k &:= m_{b_1,b_2}(\varphi_i(b_1) + n) + \varphi_k \langle \lambda, \alpha_i^\vee \rangle - \varphi_k(\varphi_k + 1)/2, \\
L_{k,\tilde{b}',\tilde{b}''} &:= d^{(\varphi_k),i}_{*\tilde{b}_2, *\tilde{b}'} f^{(\varphi_i(b_1)+n),i}_{\tilde{b}',\tilde{b}''} E^{(k),i}_{*(\tilde{b}'') \otimes t_{\lambda_1} \otimes \tilde{b}^\omega_\infty, *(\tilde{b}_1) \otimes t_{\lambda_1} \otimes \tilde{b}^\omega_\infty}.
\end{aligned}$$

The last equality follows from Remark 2.25 and Proposition 2.17. This proves the first part of the proposition.

Claim 3. $L_{k,\tilde{b}',\tilde{b}''} = 0$ *unless* $\varepsilon_i^*(\tilde{b}') \leq \varepsilon_{b_1,b_2} + k$ *and* $\varepsilon_i(\tilde{b}'') \leq \varepsilon_i(\tilde{b}_1)$. *Here* $\varepsilon_{b_1,b_2} := \min\{\varepsilon_i^*(\tilde{b}_1), -\langle \mathrm{wt}\,\tilde{b}_2, \varpi_i^\vee \rangle - \varphi_i(b_1)\}$.

Proof of Claim 3. If $E^{(k),i}_{*(\tilde{b}'') \otimes t_{\lambda_1} \otimes \tilde{b}^\omega_\infty, *(\tilde{b}_1) \otimes t_{\lambda_1} \otimes \tilde{b}^\omega_\infty} \neq 0$, then, by Corollary 2.32,

$$\begin{aligned}
&\max\{\varepsilon_i(\tilde{b}''), \varepsilon_i(b_1)\} = \varepsilon_i^*(*(\tilde{b}'') \otimes t_{\lambda_1} \otimes \tilde{b}^\omega_\infty) \\
&\leq \varepsilon_i^*(*(\tilde{b}_1) \otimes t_{\lambda_1} \otimes \tilde{b}^\omega_\infty) = \max\{\varepsilon_i(\tilde{b}_1), \varepsilon_i(b_1)\}. \qquad (3.1)
\end{aligned}$$

Note that $\varepsilon_i(b_1) = \varepsilon_i(\tilde{b}_1)$. Hence, the condition (3.1) is equivalent to $\varepsilon_i(\tilde{b}'') \leq \varepsilon_i(\tilde{b}_1)$.

Moreover, if $E^{(k),i}_{*(\tilde{b}'') \otimes t_{\lambda_1} \otimes \tilde{b}^\omega_\infty, *(\tilde{b}_1) \otimes t_{\lambda_1} \otimes \tilde{b}^\omega_\infty} \neq 0$, then, by Corollary 2.15, Proposition 2.26 and $\varepsilon_i(b_1) = \varepsilon_i(\tilde{b}_1)$,

$$\begin{aligned}
\varphi_i^*(\tilde{b}_1) + \varepsilon_i(\tilde{b}_1) &= \varphi_i(*(\tilde{b}_1) \otimes t_{\lambda_1} \otimes \tilde{b}^\omega_\infty) \\
&\geq \varphi_i(*(\tilde{b}'') \otimes t_{\lambda_1} \otimes \tilde{b}^\omega_\infty) + k \\
&= \max\{\varphi_i^*(\tilde{b}'') + \varepsilon_i(\tilde{b}_1), 0\} + k (\geq \varphi_i^*(\tilde{b}'') + \varepsilon_i(\tilde{b}_1) + k)
\end{aligned}$$

and $\mathrm{wt}\,\tilde{b}'' + k\alpha_i = \mathrm{wt}\,\tilde{b}_1$. These conditions imply $\varepsilon_i^*(\tilde{b}_1) + k \geq \varepsilon_i^*(\tilde{b}'')$.

If $f^{(\varphi_i(b_1)+n),i}_{\tilde{b}',\tilde{b}''} \neq 0$, then, by Corollary 2.29, $\varepsilon_i^*(\tilde{b}'') \geq \varepsilon_i^*(\tilde{b}')$ and $\mathrm{wt}\,\tilde{b}'' = \mathrm{wt}\,\tilde{b}' - (\varphi_i(b_1) + n)\alpha_i$.

Hence, if $f^{(\varphi_i(b_1)+n),i}_{\tilde{b}',\tilde{b}''} E^{(k),i}_{*(\tilde{b}'')\otimes t_{\lambda_1}\otimes \tilde{b}^{\omega}_{\infty}, *(\tilde{b}_1)\otimes t_{\lambda_1}\otimes \tilde{b}^{\omega}_{\infty}} \neq 0$, then $\varepsilon_i^*(\tilde{b}') \leq \varepsilon_i^*(\tilde{b}_1) + k$ and $\mathrm{wt}\,\tilde{b}' = \mathrm{wt}\,\tilde{b}_2 + (\varphi_i(b_1) - k)\alpha_i$. These conditions imply $\varepsilon_i^*(\tilde{b}') \leq \varepsilon_{b_1,b_2} + k$. Note that $\varepsilon_i^*(\tilde{b}') \leq -\langle \mathrm{wt}\,\tilde{b}', \varpi_i^\vee\rangle$. □

Applying Proposition 2.26 and Claim 3, we have

$$\begin{aligned}
d^{(\varphi_k),i}_{*\tilde{b}_2,*\tilde{b}'} &\in q_i^{-\varphi_k\varepsilon_i^*(\tilde{b}')-\frac{1}{2}\varphi_k(\varphi_k-1)}\mathbb{Z}[q] \subset q_i^{-\varphi_k(\varepsilon_{b_1,b_2}+k)-\frac{1}{2}\varphi_k(\varphi_k-1)}\mathbb{Z}[q],\\
f^{(\varphi_i(b_1)+n),i}_{\tilde{b}',\tilde{b}''} &\in q_i^{-(\varphi_i(b_1)+n)(\varepsilon_i(\tilde{b}'')-\varphi_i(b_1)-n)}\mathbb{Z}[q] \quad \subset q_i^{-(\varphi_i(b_1)+n)(\varepsilon_i(b_1)-\varphi_i(b_1)-n)}\mathbb{Z}[q]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad = q_i^{-m_{b_1,b_2}(\varphi_i(b_1)+n)}\mathbb{Z}[q],\\
E^{(k),i}_{*(\tilde{b}'')\otimes t_{\lambda_1}\otimes\tilde{b}^{\omega}_{\infty},*b_1} &\in q_i^{-k(\varphi_i^*(b_1)-k)}\mathbb{Z}[q] = q_i^{-k(\varepsilon_i^*(\tilde{b}_1)+\varphi_k-\langle\lambda,\alpha_i^\vee\rangle)}\mathbb{Z}[q]
\end{aligned}$$

for the factors of every nonzero $L_{k,\tilde{b}',\tilde{b}''}$. These estimates prove the remaining part of the proposition. □

3.2. The involution ψ^*.

Definition 3.14. We have the $\mathbb{Q}(q)$-algebra involution $\psi^* : U^* \to U^*$ given by $f \mapsto f \circ \psi$.

Example 3.15. In $A_q[\mathfrak{sl}_2]$, we have

$$\psi^*(c_{11}) = c_{11}, \psi^*(c_{12}) = c_{21}, \psi^*(c_{21}) = c_{12}, \psi^*(c_{22}) = c_{22}.$$

Remark 3.16. The map ψ^* induces the $\mathbb{Q}(q)$-algebra involution $A_i \to A_i$ for $i \in I$, which is also denoted by ψ^*.

Remark 3.17. Let $\lambda \in P_+ \cup (-P_+)$. For $v, v' \in V(\lambda)$, we have $\psi^*(c^\lambda_{v^*,v'}) = c^\lambda_{v'^*,v}$.

Definition 3.18. For $i \in I$, we define an A_i-module $V_i' := \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathbb{Q}(q)\langle m|_i$ by

$$\begin{aligned}
c_{11}. &: \langle m|_i \longmapsto \begin{cases} 0 & \text{if } m = 0,\\ \langle m-1|_i & \text{if } m \in \mathbb{Z}_{>0},\end{cases}\\
c_{12}. &: \langle m|_i \longmapsto q_i^m\langle m|_i \qquad \text{for } m \in \mathbb{Z}_{\geq 0},\\
c_{21}. &: \langle m|_i \longmapsto -q_i^{m+1}\langle m|_i \qquad \text{for } m \in \mathbb{Z}_{\geq 0},\\
c_{22}. &: \langle m|_i \longmapsto (1-q_i^{2(m+1)})\langle m+1|_i \qquad \text{for } m \in \mathbb{Z}_{\geq 0}.
\end{aligned}$$

The following proposition immediately follows from the definition of V_i'.

Proposition 3.19. *For $i \in I$, we denote by $V_i^{\psi^*}$ the A_i-module which corresponds to the $\mathbb{Q}(q)$-algebra homomorphism $\pi_i \circ \psi^* : A_i \to \mathrm{End}_{\mathbb{Q}(q)}(V_i)$.*

Then the $\mathbb{Q}(q)$-linear map $V_i^{\psi^*} \to V_i'$, $|m\rangle_i \mapsto \langle m|_i$ is an isomorphism of A_i-modules.

Definition 3.20. We have the $\mathbb{Q}(q)$-anti-algebra involution $(\psi \circ S)^* : U^* \to U^*$ given by $F \mapsto F \circ \psi \circ S$. Note that $\psi \circ S$ is the $\mathbb{Q}(q)$-algebra, anti-coalgebra involution given by $E_i \mapsto -q_i^{-1}F_i$, $F_i \mapsto -q_iE_i$ and $K_h \mapsto K_{-h}$ ($i \in I, h \in P^\vee$).

Example 3.21. In $A_q[\mathfrak{sl}_2]$, we have

$$(\psi\circ S)^*(c_{11}) = c_{22}, (\psi\circ S)^*(c_{12}) = -q_i c_{21}, (\psi\circ S)^*(c_{21}) = -q_i^{-1}c_{12}, (\psi\circ S)^*(c_{22}) = c_{11}.$$

Remark 3.22. The map $(\psi\circ S)^*$ induces the $\mathbb{Q}(q)$-anti-algebra involution $A_i \to A_i$ for $i\in I$, which is also denoted by $(\psi\circ S)^*$.

Definition 3.23. We define the $\mathbb{Q}(q)$-bilinear form $(\ ,\ )_i : V_i\times V_i \to \mathbb{Q}(q)$ by

$$(|c\rangle_i, |c'\rangle_i)_i = \delta_{c,c'}\prod_{k=1}^{c}(1-q_i^{2k})^{-1}.$$

Then this form satisfies $(C.\Lambda, \Lambda')_i = (\Lambda, (\psi\circ S)^*(C).\Lambda')_i$ for any $C\in A_i$ and $\Lambda, \Lambda'\in V_i$, which can be checked directly using Example 3.21.

The A_i-module V_i' is the dual module of V_i in the following sense.

Proposition 3.24. *Set $V_i^\star := \bigoplus_{m\in\mathbb{Z}_{\geq 0}}\mathbb{Q}(q)(|m\rangle_i, -)_i (\subset V_i^*)$. Define the left A_i-module structure on V_i^* by $\langle C.\Xi, \Lambda\rangle = \langle \Xi, (C\circ S).\Lambda\rangle$ for $C\in A_i$, $\Xi\in V_i^*$ and $\Lambda\in V_i$. Then, $V_i^\star$ is an A_i-submodule of V_i^* and the $\mathbb{Q}(q)$-linear map $V_i'\to V_i^\star, \langle m|_i \mapsto (|m\rangle_i, -)_i$ is an isomorphism of A_i-modules.*

4. The structure of the quantized coordinate algebras

In this section, we fix the definition of the quantized coordinate algebra $A_q[\mathfrak{g}]^+$ and the embedding of the algebras $\tilde{\Omega} : A_q[\mathfrak{g}]^+ \to \check{U}^{\leq 0}\otimes\check{U}^{\geq 0}$. Such an embedding is written in the reference [3, Chapter 9]. (The conventions are slightly different from ours.) We describe the U-bimodule structure on $A_q[\mathfrak{g}]^+$ in terms of $\check{U}^{\leq 0}\otimes\check{U}^{\geq 0}$ (Proposition 4.17).

Definition 4.1. The quantized coordinate algebra $A_q[\mathfrak{g}]^+$ (resp. $A_q[\mathfrak{g}]^-$) of $\mathfrak{g}$ is the $\mathbb{Q}(q)$-vector subspace of U^* spanned by the elements

$$\{c^\lambda_{f,v} \mid f\in V(\lambda)^\star, v\in V(\lambda) \text{ and } \lambda\in P_+ (\text{resp. } \lambda\in -P_+)\}.$$

Then, $A_q[\mathfrak{g}]^\pm$ is a subalgebra of U^* and isomorphic to $\bigoplus_{\lambda\in\pm P_+} V(\lambda)^\star\otimes V(\lambda)$ as a U-bimodule. (See [6, Chapter 7].)

Remark 4.2. If $\mathfrak{g}$ is of finite type, $A_q[\mathfrak{g}]^+ = A_q[\mathfrak{g}]^-(=: A_q[\mathfrak{g}])$ and it has a Hopf algebra structure induced from that of U.

Remark 4.3. We have the $\mathbb{Q}(q)$-anti-algebra involution $\omega^* : U^*\to U^*$ given by $f\mapsto f\circ\omega$. Then, this map induces $\omega^* : A_q[\mathfrak{g}]^\pm \to A_q[\mathfrak{g}]^\mp$. In particular, $A_q[\mathfrak{g}]^-$ is the opposite algebra of $A_q[\mathfrak{g}]^+$.

From now on, we mainly deal with $A_q[\mathfrak{g}]^+$. (The algebra $A_q[\mathfrak{g}]^-$ will be used in Subsection 5.2.)

Definition 4.4. Define the $\mathbb{Q}(q)$-algebra homomorphism $R_+ (\text{resp. } R_-) : U^* \to (U^{\geq 0})^* (\text{resp. } (U^{\leq 0})^*)$ by $f\mapsto f\,|_{U^{\geq 0}}$ (resp. $f\,|_{U^{\leq 0}}$).
Set $A_q[\mathfrak{b}]^+ := R_+(A_q[\mathfrak{g}]^+)$ and $A_q[\mathfrak{b}^-]^+ := R_-(A_q[\mathfrak{g}]^+)$.

Definition 4.5. Let $\check{U}$ be a variant of the quantized enveloping algebra whose generators of its Cartan part are indexed by the elements of P (denoted by $\{K_\lambda\}_{\lambda\in P}$). (That is, $\check{U}$ has the relations $K_\lambda K_{\lambda'} = K_{\lambda+\lambda'}, K_\lambda E_i = q^{(\lambda,\alpha_i)}E_iK_\lambda$ etc. The element K_i corresponds to K_{α_i}.) The $\mathbb{Q}(q)$-algebra $\check{U}$ has a Hopf algebra structure similar to the one of U.

Moreover, we define the subalgebras $\check{U}^{\geq 0}$, $\check{U}^{\leq 0}$ etc. and the weight spaces $\check{U}_\alpha$ of $\check{U}$ similarly to $U^{\geq 0}$, $U^{\leq 0}$, U_α etc. Note that $\check{U}^+$ (resp. $\check{U}^-$) can be naturally identified with U^+ (resp. U^-).

Definition 4.6 (Drinfeld pairing). There uniquely exists a $\mathbb{Q}(q)$-bilinear map $(\ ,\ )^+ : U^{\geq 0}\times \check{U}^{\leq 0} \to \mathbb{Q}(q)$ such that

(i) $(\Delta(X), Y_1\otimes Y_2)^+ = (X, Y_1Y_2)^+$ for $X\in\check{U}^{\geq 0}, Y_1, Y_2\in\check{U}^{\leq 0}$,
(ii) $(X_2\otimes X_1, \Delta(Y))^+ = (X_1X_2, Y)^+$ for $X_1, X_2\in\check{U}^{\geq 0}, Y\in\check{U}^{\leq 0}$,
(iii) $(E_i, K_\lambda)^+ = (K_h, F_i)^+ = 0$ for $i\in I$ and $\lambda\in P, h\in P^\vee$,
(iv) $(K_h, K_\lambda)^+ = q^{-\langle\lambda,h\rangle}$ for $\lambda\in P, h\in P^\vee$,
(v) $(E_i, F_j)^+ = -\delta_{ij}\dfrac{1}{q_i-q_i^{-1}}$ for $i, j\in I$.

We also define the $\mathbb{Q}(q)$-bilinear map $(\ ,\ )^- : \check{U}^{\geq 0}\times U^{\leq 0} \to \mathbb{Q}(q)$ in the same way. (For example, $(E_i, K_h)^- = (K_\lambda, F_i)^- = 0, (K_\lambda, K_h)^- = q^{-\langle\lambda,h\rangle}$ etc.)

These bilinear form has the following properties:

- For $\lambda\in P, h\in P^\vee$ and $X\in U^+, Y\in U^-$, we have $(K_hX, K_\lambda Y)^+ = q^{-\langle\lambda,h\rangle}(X,Y)^+$ and $(K_\lambda X, K_hY)^- = q^{-\langle\lambda,h\rangle}(X,Y)^-$.
- For $\alpha,\beta\in Q_+$, $(\ ,\ )^\pm|_{U^+_\alpha\times U^-_{-\beta}} = 0$ unless $\alpha=\beta$.
- $(\ ,\ )^\pm|_{U^+_\alpha\times U^-_{-\alpha}}$ is nondegenerate.

Define the $\mathbb{Q}(q)$-linear maps as follows;

$$\begin{aligned}\Phi_+ &: \check{U}^{\leq 0}\to (U^{\geq 0})^*, \quad Y\mapsto (-,Y)^+,\\ \Phi_- &: \check{U}^{\geq 0}\to (U^{\leq 0})^*, \quad X\mapsto (X,-)^-.\end{aligned}$$

Then, Φ_+ is an injective $\mathbb{Q}(q)$-algebra homomorphism and Φ_- is an injective $\mathbb{Q}(q)$-anti-algebra homomorphism.

Define the $\mathbb{Q}(q)$-algebra isomorphisms by

$$\begin{aligned}\Psi_+ &:= \omega\circ\Phi_+^{-1} : \Phi_+(\check{U}^{\leq 0})\to\check{U}^{\geq 0},\\ \Psi_- &:= \psi\circ\Phi_-^{-1} : \Phi_-(\check{U}^{\geq 0})\to\check{U}^{\leq 0}.\end{aligned}$$

Here, the maps ω, ψ on $\check{U}$ is defined similarly to the ones on U. See Definition 2.4.

Definition 4.7. Set

$$\begin{aligned}(\ ,\ )_{\mathrm{pos}} &:= ((\psi\circ\omega)(\ ), \omega(\ ))^\pm|_{U^+\times U^+} : U^+\times U^+\to\mathbb{Q}(q),\\ (\ ,\ )_{\mathrm{neg}} &:= (\psi(\ ),\ )^\pm|_{U^-\times U^-} : U^-\times U^-\to\mathbb{Q}(q).\end{aligned}$$

Then, $(\ ,\ )_{\mathrm{pos}}$ and $(\ ,\ )_{\mathrm{neg}}$ satisfies the following properties:

- $(\ ,\ )_{\mathrm{pos}} = (\omega(\ ), \omega(\ ))_{\mathrm{neg}}$, $(1,1)_{\mathrm{pos}} = (1,1)_{\mathrm{neg}} = 1$.
- $(E_iX_1, X_2)_{\mathrm{pos}} = \dfrac{1}{1-q_i^2}(X_1, \overline{f_i'(\overline{X_2})})_{\mathrm{pos}}$, $(X_1E_i, X_2)_{\mathrm{pos}} = \dfrac{1}{1-q_i^2}(X_1, \overline{{}_if'(\overline{X_2})})_{\mathrm{pos}}$.
- $(F_iY_1, Y_2)_{\mathrm{neg}} = \dfrac{1}{1-q_i^2}(Y_1, e_i'(Y_2))_{\mathrm{neg}}$, $(Y_1F_i, Y_2)_{\mathrm{neg}} = \dfrac{1}{1-q_i^2}(Y_1, {}_ie'(Y_2))_{\mathrm{neg}}$.

- $(\ ,\ )_{\mathrm{pos}}$ and $(\ ,\ )_{\mathrm{neg}}$ are symmetric.
- $(\ ,\ )_{\mathrm{pos}}\ |_{U_\alpha^+\times U_\beta^+}= 0$ and $(\ ,\ )_{\mathrm{neg}}\ |_{U_{-\alpha}^-\times U_{-\beta}^-}= 0$ unless $\alpha=\beta$. $(\alpha,\beta\in Q_+)$
- $(\ ,\ )_{\mathrm{pos}}\ |_{U_\alpha^+\times U_\alpha^+}$ and $(\ ,\ )_{\mathrm{neg}}\ |_{U_{-\alpha}^-\times U_{-\alpha}^-}$ are nondegenerate. $(\alpha\in Q_+)$

By the fact that $\dim U^\pm_{\pm\alpha}<\infty$ for all $\alpha\in Q_+$ and the nondegeneracy of the forms, there exists a basis $\{G^\pm(\tilde{b})^\vee\}_{\tilde{b}\in B(\mp\infty)}$ of $U^\pm$ such that

$$(G^+(\tilde{b})^\vee, G^+(\tilde{b}'))_{\mathrm{pos}}=\delta_{\tilde{b},\tilde{b}'}\ (\tilde{b},\tilde{b}'\in B(-\infty)).$$
$$(\text{resp. } (G^-(\tilde{b})^\vee, G^-(\tilde{b}'))_{\mathrm{neg}}=\delta_{\tilde{b},\tilde{b}'}\ (\tilde{b},\tilde{b}'\in B(\infty)).)$$

The basis $\{G^\pm(\tilde{b})^\vee\}_{\tilde{b}\in B(\mp\infty)}$ is called the upper global basis of $U^\pm$ (with respect to $(\ ,\ )_{\mathrm{pos}}$ (resp. $(\ ,\ )_{\mathrm{neg}}$)). Note that $\omega(G^-(\tilde{b})^\vee)=G^+(\tilde{b}^\omega)^\vee$ for $\tilde{b}\in B(\infty)$.

Lemma 4.8. *Let $F\in U^*$ be an element such that $F.K_h=q^{\langle\lambda,h\rangle}F$ and $K_h.F=q^{\langle\mu,h\rangle}F$ for all $h\in P^\vee$ and some $\lambda,\mu\in P$. Then, we have $R_+(F)\in\Phi_+(\check{U}^{\leq 0})$ and $R_-(F)\in\Phi_-(\check{U}^{\geq 0})$. In particular, $A_q[\mathfrak{b}^+]^+\subset\Phi_+(\check{U}^{\leq 0})$ and $A_q[\mathfrak{b}^-]^+\subset\Phi_-(\check{U}^{\geq 0})$.*

Proof. For any $G\in U_\alpha(\alpha\in Q)$, we have $\langle F,G\rangle=0$ unless $\alpha=\lambda-\mu$.

Since $(\ ,\ )^+\ |_{U^+_{\lambda-\mu}\times U^-_{-(\lambda-\mu)}}$ is nondegenerate, there uniquely exists $G'\in U^-_{-(\lambda-\mu)}$ such that $(G,G')^+=\langle F,G\rangle$ for all $G\in U^+_{\lambda-\mu}$. Hence, for $h\in P^\vee$ and $G\in U^+$, we have $\langle R_+(F),K_hG\rangle = q^{\langle\lambda,h\rangle}\langle F,G\rangle=q^{\langle\lambda,h\rangle}(G,G')^+=(K_hG,K_{-\lambda}G')^+=\langle\Phi_+(K_{-\lambda}G'),K_hG\rangle$. We can show $R_-(F)\in\Phi_-(\check{U}^{\geq 0})$ similarly. □

The following lemma is straightforward.

Lemma 4.9. *Let F be an element of $\Phi_+(\check{U}^{\leq 0})$ satisfying $F.K_h=q^{\langle\lambda,h\rangle}F$. Then $\Psi_+(F)=XK_\lambda$ where X is a unique element of U^+ such that $(X,-)_{\mathrm{pos}}=F\circ\psi\circ\omega\ |_{U^+}$. That is, $X=\sum_{\tilde{b}\in B(\infty)}\langle F,\psi(G^-(\tilde{b}))\rangle G^+(\tilde{b}^\omega)^\vee$.*

Let F' be an element of $\Phi_-(\check{U}^{\geq 0})$ satisfying $K_h.F'=q^{\langle\lambda,h\rangle}F'$. Then $\Psi_-(F')=YK_{-\lambda}$ where Y is a unique element of U^- such that $(Y,-)_{\mathrm{neg}}=F'\ |_{U^-}$. That is, $Y=\sum_{\tilde{b}\in B(\infty)}\langle F',G^-(\tilde{b})\rangle G^-(\tilde{b})^\vee$.

In particular, for $\lambda\in P_+$ and $b\in B(\lambda)$, we have

$$\Psi_+(R_+(c^\lambda_{f_\lambda,g_b^\vee}))=G^+(\tilde{b}^\omega)^\vee K_\lambda \text{ and } \Psi_-(R_-(c^\lambda_{(g_b^\vee)^*,v_\lambda}))=G^-(\tilde{b})^\vee K_{-\lambda}.$$

Here, $\tilde{b}$ is an element of $B(\infty)$ such that $G^-(\tilde{b}).v_\lambda=g_b$.

We have the injective $\mathbb{Q}(q)$-linear map $\Omega: U^*\to(U^{\leq 0}\otimes U^{\geq 0})^*$ dual to the surjective multiplication map $U^{\leq 0}\otimes U^{\geq 0}\to U$. Then, Ω is a $\mathbb{Q}(q)$-algebra homomorphism. The algebra $A_q[\mathfrak{b}^-]^+\otimes A_q[\mathfrak{b}^+]^+$ is regarded as a subalgebra of $(U^{\leq 0}\otimes U^{\geq 0})^*$.

Lemma 4.10. *We have $\Omega(A_q[\mathfrak{g}]^+)\subset A_q[\mathfrak{b}^-]^+\otimes A_q[\mathfrak{b}^+]^+$.*

Proof. For $\lambda\in P_+, f\in V(\lambda)^\star, v\in V(\lambda)$, we have

$$\Omega(c^\lambda_{f,v})=\sum\nolimits_{b\in B(\lambda)}R_-(c^\lambda_{f,g_b})\otimes R_+(c^\lambda_{f_b,v}). \tag{4.1}$$

Since the weights of $V(\lambda)$ are contained in $\lambda-Q_+$ and all weights spaces are finite dimensional, the above summation is finite. □

Proposition 4.11. *The composition* $\tilde{\Omega} : A_q[\mathfrak{g}]^+ \xrightarrow{\Omega} A_q[\mathfrak{b}^-]^+ \otimes A_q[\mathfrak{b}^+]^+ \xrightarrow{\Psi_- \otimes \Psi_+} \check{U}^{\leq 0} \otimes \check{U}^{\geq 0}$ *is a well-defined injective* $\mathbb{Q}(q)$*-algebra homomorphism. Moreover, the following hold:*

(i) *The multiplicative subset* $\mathcal{S} := \{c^{\lambda}_{f_\lambda, v_\lambda}\}_{\lambda \in P_+}$ *of* $A_q[\mathfrak{g}]^+$ *consists of non-zero divisors and it is a left and right Ore set, that is,* $\mathcal{S}\cdot C \cap A_q[\mathfrak{g}]^+ \cdot c^{\lambda}_{f_\lambda, v_\lambda} \neq \emptyset$ *and* $C \cdot \mathcal{S} \cap c^{\lambda}_{f_\lambda, v_\lambda} \cdot A_q[\mathfrak{g}]^+ \neq \emptyset$ *for all* $C \in A_q[\mathfrak{g}]^+$ *and* $\lambda \in P_+$*; hence, we can consider the algebra of fractions* $A_q[\mathfrak{g}]^+_{\mathcal{S}}$ *with respect to* $\mathcal{S}$*. See, for instance,* [3, Appendix A.2].

(ii) *The map* $\tilde{\Omega}$ *can be extended to the injective* $\mathbb{Q}(q)$*-algebra homomorphism* $A_q[\mathfrak{g}]^+_{\mathcal{S}} \to \check{U}^{\leq 0} \otimes \check{U}^{\geq 0}$ *and* $\tilde{\Omega}(A_q[\mathfrak{g}]^+_{\mathcal{S}}) = \bigoplus_{\lambda \in P} U^- K_{-\lambda} \otimes U^+ K_\lambda$*.*

Proof. The first half of the statement follows from Lemma 4.8 and 4.10. Hence, we prove (i) and (ii) from now on. By Lemma 4.9 and the calculation (4.1), we have

$$\tilde{\Omega}(c^{\lambda}_{(g_b^\vee)^*, v_\lambda}) = G^-(\tilde{b})^\vee K_{-\lambda} \otimes K_\lambda, \text{ and } \tilde{\Omega}(c^{\lambda}_{f_\lambda, g_b^\vee}) = K_{-\lambda} \otimes G^+(\tilde{b}^\omega)^\vee K_\lambda. \tag{4.2}$$

for $\lambda \in P_+$, $b \in B(\lambda)$ and $\tilde{b} \in B(\infty)$ with $G^-(\tilde{b}).v_\lambda = g_b$. In particular, $\tilde{\Omega}(c^{\lambda}_{f_\lambda, v_\lambda}) = K_{-\lambda} \otimes K_\lambda$. Therefore, the elements of $\mathcal{S}$ are non-zero divisors. For $i \in I$,

$$\begin{aligned}
\tilde{\Omega}(c^{\varpi_i}_{f_{s_i\varpi_i}, v_{\varpi_i}}) &= (F_i)^\vee K_{-\varpi_i} \otimes K_{\varpi_i} = (1 - q_i^2) F_i K_{-\varpi_i} \otimes K_{\varpi_i}, \\
\tilde{\Omega}(c^{\varpi_i}_{f_{\varpi_i}, v_{s_i\varpi_i}}) &= K_{-\varpi_i} \otimes (E_i)^\vee K_{\varpi_i} = (1 - q_i^2) K_{-\varpi_i} \otimes E_i K_{\varpi_i}.
\end{aligned}$$

Hence, for $X \in U^+$ and $Y \in U^-$, we have $YK_{-\nu} \otimes XK_\nu \in \tilde{\Omega}(A_q[\mathfrak{g}]^+)$ whenever $\langle \nu, \alpha_i^\vee \rangle$'s are sufficiently large for all $i \in I$.

Take an arbitrary $\lambda \in P_+$ and an arbitrary $C \in A_q[\mathfrak{g}]^+$. By Lemma 4.9, we write $\tilde{\Omega}(C) = \sum_{\mu \in P, \alpha, \beta \in Q_+} Y_{-\beta,\mu} K_{-\mu} \otimes X_{\alpha,\mu} K_\mu$ with $X_{\alpha,\mu} \in U^+_\alpha$ and $Y_{-\beta,\mu} \in U^-_{-\beta}$. Then,

$$\tilde{\Omega}(c^{\lambda}_{f_\lambda, v_\lambda}) \tilde{\Omega}(C) = (\sum\nolimits_{\mu \in P, \alpha, \beta \in Q_+} q^{(\lambda, \alpha+\beta)} Y_{-\beta,\mu} K_{-\mu} \otimes X_{\alpha,\mu} K_\mu) K_{-\lambda} \otimes K_\lambda$$

It follows from the above argument that $Y_{-\beta,\mu} K_{-\nu} \otimes X_{\alpha,\mu} K_\nu \in \tilde{\Omega}(A_q[\mathfrak{g}]^+)$ for all $\mu \in P$ and $\alpha, \beta \in Q_+$ whenever $\langle \nu, \alpha_i^\vee \rangle$'s are sufficiently large for all $i \in I$. Fix such an element $\nu_0 \in P_+$. Then,

$$\tilde{\Omega}(c^{\lambda+\nu_0}_{f_{\lambda+\nu_0}, v_{\lambda+\nu_0}}) \tilde{\Omega}(C) = \tilde{\Omega}(c^{\nu_0}_{f_{\nu_0}, v_{\nu_0}}) \tilde{\Omega}(c^{\lambda}_{f_\lambda, v_\lambda}) \tilde{\Omega}(C) \in \tilde{\Omega}(A_q[\mathfrak{g}]^+) \tilde{\Omega}(c^{\lambda}_{f_\lambda, v_\lambda}).$$

Hence, $\mathcal{S} \cdot C \cap A_q[\mathfrak{g}]^+ \cdot c^{\lambda}_{f_\lambda, v_\lambda} \neq \emptyset$. Similarly, we can prove $C \cdot \mathcal{S} \cap c^{\lambda}_{f_\lambda, v_\lambda} \cdot A_q[\mathfrak{g}]^+ \neq \emptyset$. This proves (i).

Since the elements $\tilde{\Omega}(c^{\lambda}_{f_\lambda, v_\lambda}) = K_{-\lambda} \otimes K_\lambda$ are invertible in $\check{U}^{\leq 0} \otimes \check{U}^{\geq 0}$, the $\mathbb{Q}(q)$-algebra homomorphism $\tilde{\Omega}$ can be extended to the injective $\mathbb{Q}(q)$-algebra homomorphism $A_q[\mathfrak{g}]^+_{\mathcal{S}} \to \check{U}^{\leq 0} \otimes \check{U}^{\geq 0}$. By (4.2), we have $\tilde{\Omega}(A_q[\mathfrak{g}]^+_{\mathcal{S}}) \supset \bigoplus_{\lambda \in P} U^- K_{-\lambda} \otimes U^+ K_\lambda$. On the other hand, by Lemma 4.9 and the calculation (4.1), we have $\tilde{\Omega}(A_q[\mathfrak{g}]^+_{\mathcal{S}}) \subset \bigoplus_{\lambda \in P} U^- K_{-\lambda} \otimes U^+ K_\lambda$. This proves (ii). □

Definition 4.12. For $w \in W$, we define the $\mathbb{Q}(q)$-linear subspace $A_q[\mathfrak{g}]^{w(\mathrm{hi})}$ and ${}^{w(\mathrm{hi})}A_q[\mathfrak{g}]$ of $A_q[\mathfrak{g}]^+$ by

$$\begin{aligned}
{}^{w(\mathrm{hi})}A_q[\mathfrak{g}] &:= \mathrm{span}_{\mathbb{Q}(q)}\{c^{\lambda}_{f_{w\lambda}, v} \mid v \in V(\lambda), \lambda \in P_+\}, \\
A_q[\mathfrak{g}]^{w(\mathrm{hi})} &:= \mathrm{span}_{\mathbb{Q}(q)}\{c^{\lambda}_{f, v_{w\lambda}} \mid f \in V(\lambda)^\star, \lambda \in P_+\}.
\end{aligned}$$

Then, they are $\mathbb{Q}(q)$-subalgebras of $A_q[\mathfrak{g}]^+$. The map ψ^* induces the isomorphism ψ^* :

$A_q[\mathfrak{g}]^{w(\mathrm{hi})} \to {}^{w(\mathrm{hi})}A_q[\mathfrak{g}]$.

Corollary 4.13. *The $\mathbb{Q}(q)$-algebra $A_q[\mathfrak{g}]^+_{\mathcal{S}}$ is generated by ${}^{e(\mathrm{hi})}A_q[\mathfrak{g}]$, $A_q[\mathfrak{g}]^{e(\mathrm{hi})}$ and $\{(c^\lambda_{f_\lambda,v_\lambda})^{-1}\}_{\lambda\in P_+}$.*

Corollary 4.14. *For $\lambda \in P_+$, we set*

$$U^-(\lambda) := \bigoplus_{\substack{\tilde{b}\in B(\infty)\\ \varepsilon_i^*(\tilde{b})\leq\langle\lambda,\alpha_i^\vee\rangle \text{ for all } i\in I}} \mathbb{Q}(q)G^-(\tilde{b})^\vee \text{ and } U^+(\lambda) := \bigoplus_{\substack{\tilde{b}\in B(-\infty)\\ \varphi_i^*(\tilde{b})\leq\langle\lambda,\alpha_i^\vee\rangle \text{ for all } i\in I}} \mathbb{Q}(q)G^+(\tilde{b})^\vee.$$

Then, for $w \in W$, the $\mathbb{Q}(q)$-linear maps

$$h_+^w : {}^{w(\mathrm{hi})}A_q[\mathfrak{g}] \to \bigoplus_{\lambda\in P_+} U^+(\lambda)K_\lambda (\subset \check{U}^{\geq 0}),\ c^\lambda_{f_{w\lambda},v} \mapsto (\Psi_+ \circ R_+)(c^\lambda_{f_\lambda,v}),$$
$$h_-^w : A_q[\mathfrak{g}]^{w^{-1}(\mathrm{hi})} \to \bigoplus_{\lambda\in P_+} U^-(\lambda)K_{-\lambda} (\subset \check{U}^{\leq 0}),\ c^\lambda_{f,v_{w^{-1}\lambda}} \mapsto (\Psi_- \circ R_-)(c^\lambda_{f,v_\lambda})$$

are the isomorphisms of $\mathbb{Q}(q)$-algebras. In particular, the set ${}_w\mathcal{S} := \{c^\lambda_{f_{w\lambda},v_\lambda}\}_{\lambda\in P_+}$ (resp. $\mathcal{S}_{w^{-1}} := \{c^\lambda_{f_\lambda,v_{w^{-1}\lambda}}\}_{\lambda\in P_+}$) is a left and right Ore multiplicative set in ${}^{w(\mathrm{hi})}A_q[\mathfrak{g}]$ (resp. $A_q[\mathfrak{g}]^{w^{-1}(\mathrm{hi})}$) and the map h_+^w (resp. h_-^w) can be extended to the $\mathbb{Q}(q)$-algebra isomorphism $h_+^w : {}^{w(\mathrm{hi})}A_q[\mathfrak{g}]_{{}_w\mathcal{S}} \to \check{U}^{\geq 0}$ (resp. $h_-^w : A_q[\mathfrak{g}]^{w^{-1}(\mathrm{hi})}_{\mathcal{S}_{w^{-1}}} \to \check{U}^{\leq 0}$), which is also denoted by h_+^w (resp. h_-^w).

Proof. The statements for $h_\pm^e$ (including the fact that $\bigoplus_{\lambda\in P_+} U^\pm(\lambda)K_{\pm\lambda}$ are closed under multiplication) follow from Proposition 2.28 and the calculation (4.2). By Proposition 2.11, the following $\mathbb{Q}(q)$-linear isomorphisms are the isomorphisms of $\mathbb{Q}(q)$-algebras:

$${}^{w(\mathrm{hi})}A_q[\mathfrak{g}] \to {}^{e(\mathrm{hi})}A_q[\mathfrak{g}], c^\lambda_{f_{w\lambda},v} \mapsto \langle f_{w\lambda}, T'_{w,1}(-.v)\rangle = c^\lambda_{f_\lambda,v},$$
$$A_q[\mathfrak{g}]^{w^{-1}(\mathrm{hi})} \to A_q[\mathfrak{g}]^{e(\mathrm{hi})}, c^\lambda_{f,v_{w^{-1}\lambda}} \mapsto \langle f, -.T''_{w,-1}v_{w^{-1}\lambda})\rangle = c^\lambda_{f,v_\lambda}.$$

See also [18, Lemma 39.1.2]. These proves the statements for $h_\pm^w$ ($w \in W$). □

Lemma 4.15. *For $\lambda \in P_+$ and weight vectors $f \in V(\lambda)^\star$, $v \in V(\lambda)$, we have*

$$\tilde{\Omega}(c^\lambda_{f,v}) = \sum_{\tilde{b},\tilde{b}'\in B(\infty)} \langle f, G^-(\tilde{b})\psi(G^-(\tilde{b}')).v\rangle G^-(\tilde{b})^\vee K_{\mathrm{wt}\,\tilde{b}-\mathrm{wt}\,f} \otimes G^+(\tilde{b}'^\omega)^\vee K_{-\mathrm{wt}\,\tilde{b}'+\mathrm{wt}\,v}.$$

Proof. By Lemma 4.9, we have

$$\begin{aligned}
&\tilde{\Omega}(c^\lambda_{f,v})\\
&= \sum_{b\in B(\lambda)} (\Psi_- \circ R_-)(c^\lambda_{f,g_b}) \otimes (\Psi_+ \circ R_+)(c^\lambda_{f_b,v})\\
&= \sum_{b\in B(\lambda)} \left(\sum_{\tilde{b},\tilde{b}'\in B(\infty)} \langle f, G^-(\tilde{b}).g_b\rangle\langle f_b, \psi(G^-(\tilde{b}')).v\rangle G^-(\tilde{b})^\vee K_{-\mathrm{wt}\,b} \otimes G^+(\tilde{b}'^\omega)^\vee K_{\mathrm{wt}\,b}\right)\\
&= \sum_{\tilde{b},\tilde{b}'\in B(\infty)} \langle f, G^-(\tilde{b})\psi(G^-(\tilde{b}')).v\rangle G^-(\tilde{b})^\vee K_{\mathrm{wt}\,\tilde{b}-\mathrm{wt}\,f} \otimes G^+(\tilde{b}'^\omega)^\vee K_{-\mathrm{wt}\,\tilde{b}'+\mathrm{wt}\,v}.
\end{aligned}$$

□

Corollary 4.16. *Define $(\ )^{\mathrm{exc}} : \check{U} \otimes \check{U} \to \check{U} \otimes \check{U}$ by $Y \otimes X \mapsto X \otimes Y$. Then,*

$$\tilde{\Omega}(\psi^*(C)) = (\omega \otimes \omega)(\tilde{\Omega}(C)^{\mathrm{exc}})$$

for all $C \in A_q[\mathfrak{g}]$.

Proposition 4.17. *Let* $i \in I$, $\lambda \in P_+$ *and weight vectors* $f \in V(\lambda)^\star$, $v \in V(\lambda)$. *Write*

$$\tilde{\Omega}(c^\lambda_{f,v}) = \sum_{\mu \in P} Y_\mu K_{-\mu} \otimes X_\mu K_\mu.$$

with $Y_\mu \in U^-$ *and* $X_\mu \in U^+$. *Then,*

$$\begin{aligned}
\tilde{\Omega}(c^\lambda_{f,E_i.v}) &= \frac{q_i^{-\langle \mathrm{wt}\, v, \alpha_i^\vee\rangle - 1}}{1-q_i^2} \sum_{\mu \in P} Y_\mu K_{-\mu} \otimes \overline{f_i'(\overline{X_\mu})} K_\mu,\\
\tilde{\Omega}(c^\lambda_{f,F_i.v}) &= \sum_{\mu \in P} \frac{q_i^{-\langle \mathrm{wt}\, X_\mu, \alpha_i^\vee\rangle}}{1-q_i^2} {}_ie'(Y_\mu) K_{-\mu+\alpha_i} \otimes X_\mu K_{\mu-\alpha_i}\\
&\quad + \sum_{\mu \in P} Y_\mu K_{-\mu} \otimes (E_i X_\mu - q_i^{\langle \mathrm{wt}\, X_\mu + 2\,\mathrm{wt}\, v, \alpha_i^\vee\rangle} X_\mu E_i) K_\mu,\\
\tilde{\Omega}(c^\lambda_{f.\psi(E_i),v}) &= \frac{q_i^{-\langle \mathrm{wt}\, f, \alpha_i^\vee\rangle - 1}}{1-q_i^2} \sum_{\mu \in P} e_i'(Y_\mu) K_{-\mu} \otimes X_\mu K_\mu,\\
\tilde{\Omega}(c^\lambda_{f.\psi(F_i),v}) &= \sum_{\mu \in P} \frac{q_i^{\langle \mathrm{wt}\, Y_\mu, \alpha_i^\vee\rangle}}{1-q_i^2} Y_\mu K_{-\mu} \otimes \overline{{}_if'(\overline{X_\mu})} K_\mu\\
&\quad + \sum_{\mu \in P} (F_i Y_\mu - q_i^{\langle -\mathrm{wt}\, Y_\mu + 2\,\mathrm{wt}\, f, \alpha_i^\vee\rangle} Y_\mu F_i) K_{-\mu} \otimes X_\mu K_\mu.
\end{aligned}$$

Proof. We only prove the first two equalities since the others are similarly proved (or follow from Corollary 4.16).

Claim 4. *For* $\lambda' \in P_+$ *and weight vectors* $f' \in V(\lambda')^\star$, $v' \in V(\lambda')$, *set* $X_{f',v'} := \sum_{\tilde{b} \in B(\infty)} \langle f', \psi(G^-(\tilde{b})).v'\rangle G^+(\tilde{b}^\omega)^\vee$ *and* $Y_{f',v'} := \sum_{\tilde{b} \in B(\infty)} \langle f', G^-(\tilde{b}).v'\rangle G^-(\tilde{b})^\vee$. *Then,*

$$\begin{aligned}
\sum_{\tilde{b} \in B(\infty)} \langle f', \psi(F_i G^-(\tilde{b})).v'\rangle G^+(\tilde{b}^\omega)^\vee &= \frac{1}{1-q_i^2} \overline{f_i'(\overline{X_{f',v'}})}\\
\sum_{\tilde{b} \in B(\infty)} \langle f', \psi(e_i'(G^-(\tilde{b}))).v'\rangle G^+(\tilde{b}^\omega)^\vee &= (1-q_i^2) E_i X_{f',v'}\\
\sum_{\tilde{b} \in B(\infty)} \langle f', \psi({}_ie'(G^-(\tilde{b}))).v'\rangle G^+(\tilde{b}^\omega)^\vee &= (1-q_i^2) X_{f',v'} E_i,\\
\sum_{\tilde{b} \in B(\infty)} \langle f', G^-(\tilde{b}) F_i.v'\rangle G^-(\tilde{b})^\vee &= \frac{1}{1-q_i^2} {}_ie'(Y_{f',v'}).
\end{aligned}$$

Proof of Claim 4. Set $X'_{f',v'} := \sum_{\tilde{b} \in B(\infty)} \langle f', \psi(F_i G^-(\tilde{b})).v'\rangle G^+(\tilde{b}^\omega)^\vee$. Note that $X_{f',v'} \in U^+_{\mathrm{wt}\, f' - \mathrm{wt}\, v'}$. Then,

$$(X'_{f',v'}, -)_{\mathrm{pos}} = (X_{f',v'}, E_i -)_{\mathrm{pos}} = \frac{1}{1-q_i^2} (\overline{f_i'(\overline{X_{f',v'}})}, -)_{\mathrm{pos}}.$$

By the nondegeneracy of the form, this proves the first equality. The others are proved similarly. □

Write $\tilde{G}_{\tilde{b},\tilde{b}'} := G^-(\tilde{b})^\vee K_{\mathrm{wt}\, \tilde{b} - \mathrm{wt}\, f} \otimes G^+(\tilde{b}'^\omega)^\vee K_{-\mathrm{wt}\, \tilde{b}' + \mathrm{wt}\, v}$. Then, we have

$$\begin{aligned}\tilde{\Omega}(c^{\lambda}_{f,E_iv}) &= \sum_{\tilde{b},\tilde{b}'\in B(\infty)} \langle f, G^-(\tilde{b})\psi(G^-(\tilde{b}'))E_i.v\rangle \tilde{G}_{\tilde{b},\tilde{b}'}(1\otimes K_i)\\ &= q_i^{-1-\langle \mathrm{wt}\,v,\alpha_i^\vee\rangle}\sum_{\tilde{b},\tilde{b}'\in B(\infty)} \langle f, G^-(\tilde{b})\psi(F_iG^-(\tilde{b}')).v\rangle \tilde{G}_{\tilde{b},\tilde{b}'}(1\otimes K_i),\ \text{and}\end{aligned}$$

$$\begin{aligned}&\tilde{\Omega}(c^{\lambda}_{f,F_iv})\\ &= \sum_{\tilde{b},\tilde{b}'\in B(\infty)} \langle f, G^-(\tilde{b})\psi(G^-(\tilde{b}'))F_i.v\rangle \tilde{G}_{\tilde{b},\tilde{b}'}(1\otimes K_i^{-1})\\ &= q_i^{-1+\langle \mathrm{wt}\,v,\alpha_i^\vee\rangle}\sum_{\tilde{b},\tilde{b}'\in B(\infty)} \langle f, G^-(\tilde{b})\psi(E_iG^-(\tilde{b}')).v\rangle \tilde{G}_{\tilde{b},\tilde{b}'}(1\otimes K_i^{-1})\\ &= \sum_{\tilde{b},\tilde{b}'\in B(\infty)} q_i^{\langle \mathrm{wt}\,\tilde{b}',\alpha_i^\vee\rangle}\langle f, G^-(\tilde{b})F_i\psi(G^-(\tilde{b}')).v\rangle \tilde{G}_{\tilde{b},\tilde{b}'}(1\otimes K_i^{-1})\\ &+\sum_{\tilde{b},\tilde{b}'\in B(\infty)} \left\langle f, G^-(\tilde{b})\psi\left(\frac{q_i^{-\langle \mathrm{wt}\,\tilde{b}'-2\,\mathrm{wt}\,v,\alpha_i^\vee\rangle-2}\,{}_ie'(G^-(\tilde{b}'))-e_i'(G^-(\tilde{b}'))}{q_i^2-1}\right).v\right\rangle \tilde{G}_{\tilde{b},\tilde{b}'}(1\otimes K_i^{-1}).\end{aligned}$$

The desired equalities are obtained from Claim 4 and these equalities by direct calculation. The details are left to the reader. □

5. Representations of the quantized coordinate algebras $A_q[\mathfrak{g}]^{\pm}$

In this section, we investigate the representations of $A_q[\mathfrak{g}]^{\pm}$ using the global bases and PBW-bases. First, the transitions between these bases are studied as preliminaries. Next, we prove the main results of this paper (Theorem 5.20, 5.32). As corollaries of Theorem 5.20, we give alternative proofs of Soibelman's tensor product theorem (Corollary 5.21, Proposition 5.30) and Kuniba-Okado-Yamada and Tanisaki's common structure theorem together with Saito-Tanisaki's theorem (Corollary 5.23, Remark 5.24). The original papers of these facts are given before each statements.

5.1. The transition from the global bases to PBW-type elements.

Definition 5.1. Let $w\in W$ and $\mathbf{i}=(i_1,i_2,\dots,i_l)\in I(w)$. Set

$$\begin{aligned}E_{\mathbf{i}}^{\mathbf{c}} &:= E_{i_l}^{(c_l)}T'_{s_{i_l},1}(E_{i_{l-1}}^{(c_{l-1})})T'_{s_{i_l}s_{i_{l-1}},1}(E_{i_{l-2}}^{(c_{l-2})})\cdots T'_{s_{i_l}\cdots s_{i_2},1}(E_{i_1}^{(c_1)}),\ \text{and}\\ F_{\mathbf{i}}^{\mathbf{c}} &:= F_{i_l}^{(c_l)}T''_{s_{i_l},1}(F_{i_{l-1}}^{(c_{l-1})})T''_{s_{i_l}s_{i_{l-1}},1}(F_{i_{l-2}}^{(c_{l-2})})\cdots T''_{s_{i_l}\cdots s_{i_2},1}(F_{i_1}^{(c_1)})(=\omega(E_{\mathbf{i}}^{\mathbf{c}})),\end{aligned}$$

where $\mathbf{c}=(c_1,c_2,\dots,c_l)\in\mathbb{Z}^l_{\geq 0}$. The ordering of the powers might seem strange but this is convenient for later use. These are called PBW-type elements. In fact, $\{E_{\mathbf{i}}^{\mathbf{c}}\}_{\mathbf{c}\in\mathbb{Z}^l_{\geq 0}}$ and $\{F_{\mathbf{i}}^{\mathbf{c}}\}_{\mathbf{c}\in\mathbb{Z}^l_{\geq 0}}$ are linearly independent sets of U^+ and U^- respectively [18, Proposition 40.2.1].

Let us denote by $U^+(w)$ (resp. $U^-(w)$) the $\mathbb{Q}(q)$-vector subspace of U^+ (resp. U^-) spanned by $\{E_{\mathbf{i}}^{\mathbf{c}}\}_{\mathbf{c}\in\mathbb{Z}^l_{\geq 0}}$ (resp. $\{F_{\mathbf{i}}^{\mathbf{c}}\}_{\mathbf{c}\in\mathbb{Z}^l_{\geq 0}}$). Here $U^{\pm}(e)=\mathbb{Q}(q)$. Theses subspaces does not depend on the choice of $\mathbf{i}\in I(w)$ [18, Proposition 40.2.1]. Note that $\omega(U^-(w))=U^+(w)$. We mainly deal with $U^-(w)$ in this subsection.

It is known that the subspace $U^-(w)$ satisfies $U^-(w)=U^-\cap T''_{w^{-1},1}(U^{\geq 0})$. In particular, this is a $\mathbb{Q}(q)$-algebra.

Proposition 5.2 ([26, Lemma 2.8]). *For $w \in W$, $U^-(w)$ is a left coideal of $U^{\leq 0}$. (That is, $\Delta(U^-(w)) \subset U^{\leq 0} \otimes U^-(w)$.)*

Proposition 5.3 ([18, Proposition 38.1.6]). *For all $i \in I$,*

$$\operatorname{Ker} e_i' = \{Y \in U^- \mid {T''_{i,1}}^{-1}(Y) \in U^-\} \text{ and } \operatorname{Ker} {}_ie' = \{Y \in U^- \mid T''_{i,1}(Y) \in U^-\}.$$

Proposition 5.4 ([18, Proposition 38.2.1]). *For $Y_1, Y_2 \in \operatorname{Ker} e_i'$ ($i \in I$), we have*

$$(Y_1, Y_2)_{\mathrm{neg}} = ((T''_{i,1})^{-1}(Y_1), (T''_{i,1})^{-1}(Y_2))_{\mathrm{neg}}.$$

Proposition 5.5. *Let $w \in W$, $\mathbf{i} = (i_1, \ldots, i_l) \in I(w)$ and set*

$$\tilde{F}_{\mathbf{i}}^{\mathbf{c}} := \left(\prod_{k=1}^{l} q_{i_k}^{\frac{1}{2}c_k(c_k-1)}(1-q_{i_k}^2)^{c_k}\right) F_{i_l}^{c_l} T''_{s_{i_l},1}(F_{i_{l-1}}^{c_{l-1}}) T''_{s_{i_l}s_{i_{l-1}},1}(F_{i_{l-2}}^{c_{l-2}}) \cdots T''_{s_{i_l}\cdots s_{i_2},1}(F_{i_1}^{c_1}),$$

for $\mathbf{c} = (c_1, \ldots, c_l) \in \mathbb{Z}_{\geq 0}^l$. Then, we have

$$(\tilde{F}_{\mathbf{i}}^{\mathbf{c}}, F_{\mathbf{i}}^{\mathbf{c}'})_{\mathrm{neg}} = \delta_{\mathbf{c},\mathbf{c}'} \text{ for } \mathbf{c}, \mathbf{c}' \in \mathbb{Z}_{\geq 0}^l.$$

Proof. Using Proposition 5.3 and $e_i'(F_i^{(c)}) = q_i^{-c+1} F_i^{(c-1)}$, we have

$$(e_{i_l}')^d (F_{\mathbf{i}}^{\mathbf{c}}) = q_{i_l}^{-\frac{1}{2}d(2c_l-d-1)} F_{\mathbf{i}}^{\mathbf{c}-(0,\ldots,0,d)} \text{ for } \mathbf{c} \in \mathbb{Z}_{\geq 0}^l \text{ and } d \in \mathbb{Z}_{\geq 0},$$

where $F_{\mathbf{i}}^{\mathbf{d}-(0,\ldots,0,e)} := 0$ if $e > d_l$. Therefore, the proposition follows from the property of the bilinear form $(\ ,\ )_{\mathrm{neg}}$ and Proposition 5.4. □

Definition 5.6. By Proposition 5.5, $(\ ,\ )_{\mathrm{neg}} |_{U^-(w)_{-\alpha} \times U^-(w)_{-\alpha}}$ is nondegenerate for $\alpha \in Q_+$. Hence, we can take the orthogonal complement $U^-(w)_{-\alpha}^{\perp}$ of $U^-(w)_{-\alpha}$ in $U^-_{-\alpha}$ with $U^-_{-\alpha} = U^-(w)_{-\alpha} \oplus U^-(w)_{-\alpha}^{\perp}$. Set $U^-(w)^{\perp} := \bigoplus_{\alpha \in Q_+} U^-(w)_{-\alpha}^{\perp}$.

Let $U^+(w)^{\perp} := \omega(U^-(w)^{\perp})$. Note that $U^+(w)^{\perp}$ is the orthogonal complement of $U^+(w)$ with respect to $(\ ,\ )_{\mathrm{pos}}$.

The following corollary follows from the definition of the form $(\ ,\ )_{\mathrm{neg}}$ and Proposition 5.2.

Corollary 5.7. *For $w \in W$, $U^-(w)^{\perp}$ is a left ideal of U^-.*

Notation 5.8. Let $w \in W$. For $X \in U^{\pm}$, the image of X under the natural projection $U^{\pm} \to U^{\pm}/U^{\pm}(w)^{\perp}$ will be denoted by $[X]_w$. For $\mathbf{i} \in I(w)$, the vectors $\{[E_{\mathbf{i}}^{\mathbf{c}}]_w\}_{\mathbf{c}}$ and $\{[F_{\mathbf{i}}^{\mathbf{c}}]_w\}_{\mathbf{c}}$ form bases of $U^+/U^+(w)^{\perp}$ and $U^-/U^-(w)^{\perp}$ respectively.

Proposition 5.9. *Let $w \in W$ and fix $\mathbf{i} = (i_1, \ldots, i_l) \in I(w)$. For $\tilde{b} \in B(\infty)$, we write $[G^-(\tilde{b})]_w = \sum_{\mathbf{c} \in \mathbb{Z}_{\geq 0}^l} {}_{\mathbf{i}}\zeta_{\mathbf{c}}^{\tilde{b}} [F_{\mathbf{i}}^{\mathbf{c}}]_w$ with ${}_{\mathbf{i}}\zeta_{\mathbf{c}}^{\tilde{b}} \in \mathbb{Q}(q)$.*

Then we have

$${}_{\mathbf{i}}\zeta_{\mathbf{c}}^{\tilde{b}} = \left(\prod_{k=1}^{l} q_{i_k}^{\frac{1}{2}c_k(c_k-1)}\right) \sum_{\substack{\tilde{b}_1, \ldots, \tilde{b}_{l-1} \in B(\infty) \\ \text{with } \varepsilon_{i_k}^*(\tilde{b}_{k-1}) = 0 \text{ for all } k}} d_{\tilde{b}, \tau_{i_l}^{-1}(\tilde{b}_{l-1})}^{(c_l), i_l} d_{\tilde{b}_{l-1}, \tau_{i_{l-1}}^{-1}(\tilde{b}_{l-2})}^{(c_{l-1}), i_{l-1}} \cdots d_{\tilde{b}_1, \tilde{b}_\infty}^{(c_1), i_1}.$$

See Proposition 2.22 *for the definition of τ_i.*

Remark 5.10. For $w \in W$, $\mathbf{i} \in I(w)$ and $\tilde{b} \in B(\infty)$, we have
$$[G^{+}(\tilde{b}^{\omega})]_{w} = \sum\nolimits_{\mathbf{c}\in\mathbb{Z}^{l}_{\geq 0}} {}_{\mathbf{i}}\zeta^{\tilde{b}}_{\mathbf{c}}[E^{\mathbf{c}}_{\mathbf{i}}]_{w}.$$

Proof of Proposition 5.9. By Proposition 5.5, we have ${}_{\mathbf{i}}\zeta^{\tilde{b}}_{\mathbf{c}} = (\tilde{F}^{\mathbf{c}}_{\mathbf{i}}, G^{-}(\tilde{b}))_{\mathrm{neg}}$. Set $D := \prod_{k=1}^{l} q_{i_k}^{\frac{1}{2}c_k(c_k-1)}(1-q_{i_k}^{2})^{c_k}$. Then, we have
$$\begin{aligned}(\tilde{F}^{\mathbf{c}}_{\mathbf{i}}, G^{-}(\tilde{b}))_{\mathrm{neg}} &= D(F_{i_l}^{c_l} T''_{s_{i_l},1}(F_{i_{l-1}}^{c_{l-1}})\cdots T''_{s_{i_l}s_{i_{l-1}}\cdots s_{i_2},1}(F_{i_1}^{c_1}), G^{-}(\tilde{b}))_{\mathrm{neg}}\\ &= D(1-q_{i_l}^{2})^{-c_l}(T''_{s_{i_l},1}(F_{i_{l-1}}^{c_{l-1}})\cdots T''_{s_{i_l}s_{i_{l-1}}\cdots s_{i_2},1}(F_{i_1}^{c_1}), (e'_{i_l})^{c_l}G^{-}(\tilde{b}))_{\mathrm{neg}}\\ &= q_{i_l}^{\frac{1}{2}c_l(c_l-1)} \sum\nolimits_{\tilde{b}\in B(\infty)} d^{(c_l),i_l}_{\tilde{b},\tilde{b}'}(\tilde{F}^{(c_1,\ldots,c_{l-1},0)}_{\mathbf{i}}, G^{-}(\tilde{b}'))_{\mathrm{neg}}.\end{aligned}$$

By Proposition 2.28 (i) and Proposition 5.3, we have $(\tilde{F}^{(c_1,\ldots,c_{l-1},0)}_{\mathbf{i}}, G^{-}(\tilde{b}'))_{\mathrm{neg}} = 0$ unless $\varepsilon_{i_l}(\tilde{b}') = 0$. Moreover, when $\varepsilon_{i_l}(\tilde{b}') = 0$, by Proposition 2.22 and 5.4, we have
$$\begin{aligned}(\tilde{F}^{(c_1,\ldots,c_{l-1},0)}_{\mathbf{i}}, G^{-}(\tilde{b}'))_{\mathrm{neg}} &= (\tilde{F}^{(c_1,\ldots,c_{l-1},0)}_{\mathbf{i}}, {}^{i_l}\pi(G^{-}(\tilde{b}')))_{\mathrm{neg}}\\ &= (\tilde{F}^{(c_1,\ldots,c_{l-1},0)}_{\mathbf{i}}, T''_{i_l,1}(\pi^{i_l}(G^{-}(\tau_{i_l}(\tilde{b}')))))_{\mathrm{neg}}\\ &= (\tilde{F}^{(c_1,\ldots,c_{l-1})}_{\mathbf{i}'}, \pi^{i_l}(G^{-}(\tau_{i_l}(\tilde{b}'))))_{\mathrm{neg}}\\ &= (\tilde{F}^{(c_1,\ldots,c_{l-1})}_{\mathbf{i}'}, G^{-}(\tilde{b}_{l-1})))_{\mathrm{neg}},\end{aligned}$$

where $\mathbf{i} = (i_1, \ldots, i_{l-1})$ and $\tilde{b}_{l-1} := \tau_{i_l}(\tilde{b}')$. The last equality follows from $\tilde{F}^{(c_1,\ldots,c_{l-1})}_{\mathbf{i}'} \in \mathrm{Ker}\, {}_{i_l}e'$ (by Proposition 5.3) and $G^{-}(\tilde{b}_{l-1}) - \pi^{i_l}(G^{-}(\tilde{b}_{l-1})) \in U^{-}F_{i_l}$.

Therefore, the proposition follows by induction on l. □

Remark 5.11. By Proposition 2.26 and the proof of Proposition 5.9, we have ${}_{\mathbf{i}}\zeta^{\tilde{b}}_{\mathbf{c}} = 1$ if
$$\begin{cases} c_k = \varepsilon_{i_k}(\tau_{i_{k+1}}\tilde{e}^{\max}_{i_{k+1}}\cdots\tau_{i_l}\tilde{e}^{\max}_{i_l}\tilde{b}) \text{ for } k = 1, \ldots, l, \text{ and}\\ \tilde{b}_{\infty} = \tilde{e}^{\max}_{i_1}\tau_{i_2}\tilde{e}^{\max}_{i_2}\cdots\tau_{i_l}\tilde{e}^{\max}_{i_l}\tilde{b},\end{cases}$$

where $\tilde{e}^{\max}_{i}\tilde{b}' := \tilde{e}_{i}^{\varepsilon_i(\tilde{b}')}\tilde{b}'$ for $\tilde{b}' \in B(\infty)$, and otherwise ${}_{\mathbf{i}}\zeta^{\tilde{b}}_{\mathbf{c}} \in q\mathbb{Z}[q]$. Moreover, if there exists $\mathbf{c}_0 \in \mathbb{Z}^{l}_{\geq 0}$ such that ${}_{\mathbf{i}}\zeta^{\tilde{b}}_{\mathbf{c}_0} = 1$, then
$${}_{\mathbf{i}}\zeta^{\tilde{b}}_{\mathbf{c}} = 0 \text{ unless } \mathbf{c} \geq \mathbf{c_0}.$$

Here $\mathbf{d} = (d_1, d_2, \ldots, d_l) > \mathbf{d}' = (d'_1, d'_2, \ldots, d'_l)$ means that there exists $k \in \{1, \ldots, l\}$ such that $d_l = d'_l, \ldots, d_{k+1} = d'_{k+1}, d_k > d'_k$. This fact is known as the unitriangularity property.

It is known ([15, Theorem 11.5]) that $d^{(k),i}_{\tilde{b},\tilde{b}'} \in \mathbb{Z}_{\geq 0}[q^{\pm 1}]$ for any $\tilde{b}, \tilde{b}' \in B(\infty)$, $k \in \mathbb{Z}_{\geq 0}$ and $i \in I$ when $\mathfrak{g}$ is of symmetric Kac-Moody type. Hence, in this case, Proposition 5.9 implies that ${}_{\mathbf{i}}\zeta^{\tilde{b}}_{\mathbf{c}} \in \mathbb{Z}_{\geq 0}[q]$ for any $\tilde{b} \in B(\infty)$ and $\mathbf{c} \in \mathbb{Z}^{l}_{\geq 0}$.

Let $\mathfrak{g}$ be of finite type and w_0 the longest element of W. In this case, such positivity was originally proved by Lusztig in his original paper of the canonical bases [14, Corollary 10.7] for "adapted" elements $\mathbf{i} \in I(w_0)$ ([14, 4.7]), through his geometric realization of the elements of the canonical bases and PBW bases. More recently, this fact for arbitrary $\mathbf{i} \in I(w_0)$ was proved by Kato [9, Theorem 4.17], through the categorification of PBW bases by using the Khovanov-Lauda-Rouquier algebras. By the way, McNamara has also established the categorification of PBW bases via the Khovanov-Lauda-Rouquier algebras for arbitrary finite types [19, Theorem 3.1] (the dual PBW bases) and symmetric affine types [20, Theorem 24.4]. As a consequence, he has obtained the positivity results for symmetric affine types [20, Theorem 24.10]. Remark that our method essentially provides no data concerning “the imaginary part”.

Kimura also remarks on such positivity in [10, Remark 2.24].

5.2. The actions of $A_q[\mathfrak{g}]^+$ on their tensor product modules.

DEFINITION 5.12. For $\mathbf{i} = (i_1, \dots, i_l) \in I^l$ ($l \in \mathbb{Z}_{>0}$), we have the $\mathbb{Q}(q)$-algebra homomorphism $\hat{\Delta}_l : U^* \to (U \otimes \cdots \otimes U)^*$ (l-fold), dual to the multiplication map, and $\iota_{\mathbf{i}}^* : (U \otimes \cdots \otimes U)^* \to (U_{i_1} \otimes \cdots \otimes U_{i_l})^*$ given by the restriction. Here the $\mathbb{Q}(q)$-algebra structures of $(U \otimes \cdots \otimes U)^*$ and $(U_{i_1} \otimes \cdots \otimes U_{i_l})^*$ are induced from the natural $\mathbb{Q}(q)$-coalgebra structure of $U \otimes \cdots \otimes U$ and $U_{i_1} \otimes \cdots \otimes U_{i_l}$ respectively.

The following lemma can be checked straightforwardly.

Lemma 5.13. *Let V be an integrable U-module, $f \in V^*$, $v \in V$ and $\mathbf{i} = (i_1, \dots, i_l) \in I^l$ ($l \in \mathbb{Z}_{>0}$). Then we have $(\iota_{\mathbf{i}}^* \circ \hat{\Delta}_l)(c^V_{f,v}) \in A_{i_1} \otimes \cdots \otimes A_{i_l}$.*

DEFINITION 5.14. For $\mathbf{i} = (i_1, \dots, i_l) \in I^l$ ($l \in \mathbb{Z}_{>0}$), we define representations $\pi_{\mathbf{i}}^{\pm} : A_q[\mathfrak{g}]^{\pm} \to \mathrm{End}_{\mathbb{Q}(q)}(V_{i_1}) \otimes \cdots \otimes \mathrm{End}_{\mathbb{Q}(q)}(V_{i_l})$ by

$$\pi_{\mathbf{i}}^{\pm} := (\pi_{i_1} \otimes \cdots \otimes \pi_{i_l}) \circ \iota_{\mathbf{i}}^* \circ \hat{\Delta}_l \mid_{A_q[\mathfrak{g}]^{\pm}} .$$

These representations are well-defined by Lemma 5.13.

We write simply the corresponding $A_q[\mathfrak{g}]^{\pm}$-modules $V_{\mathbf{i}}$ or $V_{i_1} \otimes \cdots \otimes V_{i_l}$ (although the $\mathbb{Q}(q)$-algebras $A_q[\mathfrak{g}]^{\pm}$ are not bialgebras when $\mathfrak{g}$ is of infinite type). It will cause no confusion if we use the same notation for the $A_q[\mathfrak{g}]^+$-module $V_{\mathbf{i}}$ and $A_q[\mathfrak{g}]^-$-module $V_{\mathbf{i}}$. We define the $A_q[\mathfrak{g}]^{\pm}$-modules $V'_{\mathbf{i}}$, $V_{i_1} \otimes \cdots \otimes V_{i_k} \otimes V'_{j_1} \otimes \cdots \otimes V'_{j_l}$ etc. similarly.

For $\mathbf{m} = (m_1, \dots, m_l) \in \mathbb{Z}^l_{\geq 0}$, we set

$$|\mathbf{m}\rangle_{\mathbf{i}} := |m_1\rangle_{i_1} \otimes \cdots \otimes |m_l\rangle_{i_l} \in V_{\mathbf{i}} \text{ and } \langle \mathbf{m}|_{\mathbf{i}} := \langle m_1|_{i_1} \otimes \cdots \otimes \langle m_l|_{i_l} \in V'_{\mathbf{i}}.$$

Define the $A_q[\mathfrak{g}]^{\pm}$-module $V_\emptyset := \mathbb{Q}(q)|0\rangle_\emptyset$ as $C.|0\rangle_\emptyset = \langle C, 1\rangle |0\rangle_\emptyset$ for $C \in A_q[\mathfrak{g}]^{\pm}$, and denote by $\pi_\emptyset^{\pm}$ the corresponding algebra homomorphism $A_q[\mathfrak{g}]^{\pm} \to \mathrm{End}_{\mathbb{Q}(q)}(V_\emptyset)$. Set $V'_\emptyset := V_\emptyset$ and $\langle 0|_\emptyset := |0\rangle_\emptyset$.

REMARK 5.15. Henceforth all statements are obvious in the case $w = e$. Therefore we prove them only for $w \neq e$.

NOTATION 5.16. For an ordered n-tuple $\mathbf{k} = (k_1, \dots, k_n)$, set $\overline{\mathbf{k}} := (k_n, \dots, k_1)$ ($n \in \mathbb{Z}_{>0}$).

REMARK 5.17. Let $\mathbf{i} \in I^l$ ($l \in \mathbb{Z}_{>0}$). We denote by $V_{\mathbf{i}}^{\psi^*}$ the $A_q[\mathfrak{g}]^{\pm}$-module which corresponds to the $\mathbb{Q}(q)$-algebra homomorphism $\pi_{\mathbf{i}}^{\pm} \circ \psi^* : A_q[\mathfrak{g}]^{\pm} \to \mathrm{End}_{\mathbb{Q}(q)}(V_{\mathbf{i}})$.

Then, the $\mathbb{Q}(q)$-linear isomorphism $V_{\mathbf{i}}^{\psi^*} \to V'_{\overline{\mathbf{i}}}$, $|\mathbf{m}\rangle_{\mathbf{i}} \mapsto \langle \overline{\mathbf{m}}|_{\overline{\mathbf{i}}}$ is an isomorphism of $A_q[\mathfrak{g}]^{\pm}$-modules.

The $A_q[\mathfrak{g}]^{\pm}$-module $V'_{\overline{\mathbf{i}}}$, in fact, is obtained from the $A_q[\mathfrak{g}]^{\pm}$-module $V_{\overline{\mathbf{i}}}$ by a “torus element twist”, but we do not treat such a twist in this paper. See [11, Chapter 3] for details.

Lemma 5.18. *Let $w, w' \in W$ such that the length of ww' if the sum of that of w and w'. Take $\tilde{b} \in B(\infty)$. Assume that $[G^-(\tilde{b})]_w \neq 0$. Then $G^-(\tilde{b}).v_{w'\lambda} \neq 0$ whenever $\langle \lambda, \alpha_i^\vee \rangle$'s are sufficiently large for all $i \in I$.*

Proof. Let $\mathbf{i} = (i_1, \dots, i_l) \in I(w)$ and $\mathbf{i}' = (i'_1, \dots, i'_{l'}) \in I(w')$. Then,

$$\begin{aligned}
T''_{w'^{-1},1}(G^-(\tilde{b}).v_{w'\lambda}) &= T''_{w'^{-1},1}(\pi^{i'_1}(G^-(\tilde{b})).v_{w'\lambda}) \\
&= T''_{s_{i'_{l'}}\cdots s_{i'_2},1}(T''_{i'_1,1}(\pi^{i'_1}(G^-(\tilde{b}))).v_{s_{i'_2}\cdots s_{i'_{l'}}\lambda}) \\
&\cdots \\
&= T''_{i'_{l'},1}(\pi^{i'_{l'}}(T''_{i'_{l'-1},1}(\pi^{i'_{l'-1}} \cdots T''_{i'_1,1}\pi^{i'_1}(G^-(\tilde{b})) \cdots).v_\lambda
\end{aligned}$$

and $T''_{i'_{l'},1}(\pi^{i'_{l'}}(T''_{i'_{l'-1},1}(\pi^{i'_{l'-1}} \cdots T''_{i'_1,1}\pi^{i'_1}(G^-(\tilde{b})) \cdots) \in U^-$ by Proposition 5.3. On the other hand, by our assumption and Proposition 5.3, there exists $\mathbf{c} \in \mathbb{Z}^l_{\geq 0}$ such that

$$\begin{aligned}
0 &\neq (G^-(\tilde{b}), \tilde{F}^{\mathbf{c}}_{\mathbf{i}})_{\mathrm{neg}} \\
&= (\pi^{i'_1}(G^-(\tilde{b})), \tilde{F}^{\mathbf{c}}_{\mathbf{i}})_{\mathrm{neg}} \\
&= (T''_{i'_1,1}(\pi^{i'_1}(G^-(\tilde{b}))), T''_{i'_1,1}(\tilde{F}^{\mathbf{c}}_{\mathbf{i}}))_{\mathrm{neg}} \\
&\cdots \\
&= (T''_{i'_{l'},1}(\pi^{i'_{l'}}(T''_{i'_{l'-1},1}(\pi^{i'_{l'-1}} \cdots T''_{i'_1,1}(\pi^{i'_1}(G^-(\tilde{b})) \cdots), T''_{w'^{-1},1}(\tilde{F}^{\mathbf{c}}_{\mathbf{i}}))_{\mathrm{neg}}.
\end{aligned}$$

Hence, $T''_{i'_{l'},1}(\pi^{i'_{l'}}(T''_{i'_{l'-1},1}(\pi^{i'_{l'-1}} \cdots T''_{i'_1,1}(\pi^{i'_1}(G^-(\tilde{b})) \cdots) \neq 0$. This proves the lemma. □

Lemma 5.19. *Let $e \neq w \in W$, $\mathbf{i} = (i_1, \dots, i_l) \in I(w)$, $\lambda \in P_+$ and $b_0, b_l \in B(\lambda)$. Take $\tilde{b}_l \in B(\infty)$ such that $G^-(\tilde{b}_l).v_\lambda = g_{b_l}$. We can write as follows:*

$$\begin{aligned}
&\pi^+_{\mathbf{i}}(c^\lambda_{f_{b_0}, g_{b_l}}).|0\rangle_{\mathbf{i}} \\
&= \sum_{b_1,\dots,b_{l-1} \in B(\lambda)} \iota^*_{i_1}(c^\lambda_{f_{b_0}, g_{b_1}}).|0\rangle_{i_1} \otimes \cdots \otimes \iota^*_{i_k}(c^\lambda_{f_{b_{k-1}}, g_{b_k}}).|0\rangle_{i_k} \otimes \cdots \otimes \iota^*_{i_l}(c^\lambda_{f_{b_{l-1}}, g_{b_l}}).|0\rangle_{i_l}.
\end{aligned}$$

Then, in the nonzero summand of the right-hand side,

(i) $\mathrm{wt}\, b_{k-1} - \mathrm{wt}\, b_k \in \mathbb{Z}\alpha_{i_k}$ *and* $m_{b_{k-1},b_k} (:= -\langle \mathrm{wt}\, b_{k-1} + \mathrm{wt}\, b_k, \alpha^\vee_{i_k} \rangle / 2) \geq 0$,

(ii) $s_{i_l} s_{i_{l-1}} \cdots s_{i_k}(\mathrm{wt}\, b_{k-1}) = \mathrm{wt}\, b_l + \sum_{j=k}^l m_{b_{j-1},b_j} s_{i_l} s_{i_{l-1}} \cdots s_{i_{j+1}} \alpha_{i_j}$, *in particular,* $\sum_{j=k}^l m_{b_{j-1},b_j} s_{i_l} \cdots s_{i_{j+1}} \alpha_{i_j} \leq -\mathrm{wt}\, \tilde{b}_l$,

(iii) $\varphi_{i_k}(b_{k-1}) \leq -\mathrm{ht}\, \mathrm{wt}\, \tilde{b}_l$

for $k = 1, \dots, l$.

Proof. The statement (i) follows from Lemma 3.8. Set $\mathrm{wt}\, b_{k-1} = \mathrm{wt}\, b_k - n_k \alpha_{i_k}$ with $n_k \in \mathbb{Z}$. Then, $m_{b_{k-1},b_k} = -n_k - \langle \mathrm{wt}\, b_{k-1}, \alpha^\vee_{i_k} \rangle$. Hence, for $k = 1, \dots l$,

$$\begin{aligned}
s_{i_l} s_{i_{l-1}} \cdots s_{i_k}(\mathrm{wt}\, b_{k-1}) &= s_{i_l} s_{i_{l-1}} \cdots s_{i_{k+1}}(\mathrm{wt}\, b_k + m_{b_{k-1},b_k} \alpha_{i_k}) \\
&\cdots \\
&= \mathrm{wt}\, b_l + \sum_{j=k}^l m_{b_{j-1},b_j} s_{i_l} s_{i_{l-1}} \cdots s_{i_{j+1}} \alpha_{i_j}.
\end{aligned}$$

This proves (ii). Since $\mathrm{wt}\, b_{k-1} - \varphi_{i_k}(b_{k-1})\alpha_{i_k}$ is a weight of $V(\lambda)$, $s_{i_l} \cdots s_{i_k}(\mathrm{wt}\, b_{k-1}) - \varphi_{i_k}(b_{k-1}) s_{i_l} \cdots s_{i_k} \alpha_{i_k}$ is also a weight of $V(\lambda)$. Hence,

$$-Q_+ \ni s_{i_l}\cdots s_{i_k}(\mathrm{wt}\, b_{k-1}) - \varphi_{i_k}(b_{k-1})s_{i_l}\cdots s_{i_k}\alpha_{i_k} - \lambda$$
$$= \mathrm{wt}\,\tilde{b}_l + \sum\nolimits_{j=k}^{l} m_{b_{j-1},b_j} s_{i_l}s_{i_{l-1}}\cdots s_{i_j}\alpha_{i_j} + \varphi_{i_k}(b_{k-1})s_{i_l}\cdots s_{i_{k+1}}\alpha_{i_k}.$$

This proves (iii). □

The following is one of the main theorems in this paper.

Theorem 5.20. *Let $e \neq w \in W$, $\mathbf{i} = (i_1, i_2, \ldots, i_l) \in I(w)$, $\lambda \in P_+$ and $b_0, b_l \in B(\lambda)$. Take $\tilde{b}_l \in B(\infty)$ such that $G^-(\tilde{b}_l).v_\lambda = g_{b_l}$. Write*

$$\pi^+_{\mathbf{i}}(c^\lambda_{f_{b_0},g_{b_l}}).|0\rangle_{\mathbf{i}} = \sum\nolimits_{\mathbf{c}\in\mathbb{Z}^l_{\geq 0}} {}_{\mathbf{i}}\zeta^{b_0,b_l}_{\mathbf{c}}|\mathbf{c}\rangle_{\mathbf{i}}\ ({}_{\mathbf{i}}\zeta^{b_0,b_l}_{\mathbf{c}} \in \mathbb{Q}(q)).$$

Then the coefficients ${}_{\mathbf{i}}\zeta^{b_0,b_l}_{\mathbf{c}}$ are Laurent polynomials in q with integer coefficients. Moreover, when $\lambda' \in P_+$ tends to ∞ in the sense that $\langle\lambda', \alpha^\vee_i\rangle$ tends to ∞ for all $i \in I$, the coefficient ${}_{\mathbf{i}}\zeta^{v_{w\lambda'},b^{(\lambda')}_l}_{\mathbf{c}}$ with $G^-(\tilde{b}_l).v_{\lambda'} = g_{b^{(\lambda')}_l}$ converges to ${}_{\mathbf{i}}\zeta^{\tilde{b}_l}_{\mathbf{c}}$ (See Proposition 5.9) *in the complete discrete valuation field $\mathbb{Q}((q))$.*

In fact, we have more detailed results as follows; set $\lambda_{\mathbf{i},\tilde{b}_l} := \min\{(s_{i_{k+1}}\cdots s_{i_l}\lambda, \alpha_{i_k}) + 3(s_{i_{k+1}}\cdots s_{i_l}\alpha, \varpi_{i_k}) - \sum_{s=1}^{k-1}\Delta_{i_s}(\mathrm{ht}\,\mathrm{wt}\,\tilde{b}_l)^2 | \mathrm{wt}\,\tilde{b}_l \leq \alpha \leq 0, k = 1, \ldots, l\}$. Recall Notation 2.1. *Suppose that $\lambda_{\mathbf{i},\tilde{b}_l} \geq 0$. Then*

$$({}_{\mathbf{i}}\zeta^{b_0,b_l}_{\mathbf{c}})_{<\lambda_{\mathbf{i},\tilde{b}_l}} = 0 \text{ unless } g_{b_0} \in U^-.v_{w\lambda}, \text{ and}$$

if $G^-(\tilde{b}_0).v_{w\lambda} = g_{b_0} \in U^-.v_{w\lambda}$ ($\tilde{b}_0 \in B(\infty), \varepsilon^*_{i_1}(\tilde{b}_0) = 0$), we have

$$({}_{\mathbf{i}}\zeta^{b_0,b_l}_{\mathbf{c}})_{<\lambda_{\mathbf{i},\tilde{b}_l}} = \left(\left(\prod_{k=1}^{l} q_{i_k}^{\frac{1}{2}c_k(c_k-1)}\right)\sum_{(\clubsuit)} d^{(c_l),i_l}_{\tilde{b}_l,\tau^{-1}_{i_l}(\tilde{b}_{l-1})} d^{(c_{l-1}),i_{l-1}}_{\tilde{b}_{l-1},\tau^{-1}_{i_{l-1}}(\tilde{b}_{l-2})}\cdots d^{(c_1),i_1}_{\tilde{b}_1,\tau^{-1}_{i_1}(\tilde{b}_0)}\right)_{<\lambda_{\mathbf{i},\tilde{b}_l}},$$

where the summation ($\clubsuit$) runs over $b_1, \ldots, b_{l-1} \in B(\lambda)$ with $g_{b_k} = G^-(\tilde{b}_k).v_{s_{i_{k+1}}\cdots s_{i_l}\lambda}$ for some $\tilde{b}_k \in B(\infty)$ ($k = 1, \ldots, l-1$). Note that $\lambda_{\mathbf{i},\tilde{b}_l}$ goes to ∞ when λ tends to ∞ for any fixed $\tilde{b}_l$.

Proof. It follows from, for instance, Corollary 3.11 that coefficients ${}_{\mathbf{i}}\zeta^{b_0,b_l}_{\mathbf{c}}$ are Laurent polynomials in q with integer coefficients. The next assertion follows from Proposition 5.9, Lemma 5.18 and the latter half of the theorem. We compute $\pi^+_{\mathbf{i}}(c^\lambda_{f_{b_0},g_{b_l}}).|0\rangle_{\mathbf{i}}$ as in the right hand side of the equality in Lemma 5.19 and from the rightmost component. Then the desired results follow by using Corollary 3.11, Corollary 3.12, Remark 3.13 and Lemma 5.19 repeatedly. □

The following is known as Soibelman's tensor product theorem. This has been originally proved in [25] (finite case), [21], [26] (symmetrizable Kac-Moody case). Note that the quantized coordinate algebra in [21] is larger than ours. The irreducibility will be proved in Proposition 5.30.

Corollary 5.21. *Let $w \in W$. Then the isomorphism class of $V_{\mathbf{i}}$ (resp. $V'_{\mathbf{i}}$) as an $A_q[\mathfrak{g}]^+$-module does not depend on the choice of $\mathbf{i} \in I(w)$. Moreover, the $A_q[\mathfrak{g}]^+$-module $V_{\mathbf{i}}$ (resp. $V'_{\mathbf{i}}$) is generated by $|0\rangle_{\mathbf{i}}$ (resp. $\langle 0|_{\mathbf{i}}$), and, for any $\mathbf{i}, \mathbf{j} \in I(w)$, an isomorphism $V_{\mathbf{i}} \to V_{\mathbf{j}}$ (resp. $V'_{\mathbf{i}} \to V'_{\mathbf{j}}$) of $A_q[\mathfrak{g}]^+$-modules is given by $|0\rangle_{\mathbf{i}} \mapsto |0\rangle_{\mathbf{j}}$ (resp. $\langle 0|_{\mathbf{i}} \mapsto \langle 0|_{\mathbf{j}}$).*

Definition 5.22. We will identify the $A_q[\mathfrak{g}]^+$-modules $V_{\mathbf{i}}$ (resp. $V'_{\mathbf{i}}$) ($\mathbf{i} \in I(w)$) via the isomorphisms in Corollary 5.21 and write them as V_w (resp. V'_w). Denote by π_w^+ (resp. $\pi_w^{'+}$) the corresponding $\mathbb{Q}(q)$-algebra homomorphism $A_q[\mathfrak{g}]^+ \to \mathrm{End}_{\mathbb{Q}(q)}(V_w)$(resp. $\mathrm{End}_{\mathbb{Q}(q)}(V'_w)$) and by $|0\rangle_w$ (resp. $\langle 0|_w$) the vector $|0\rangle_{\mathbf{i}}$ (resp. $\langle 0|_{\mathbf{i}}$).

Proof of Corollary 5.21. The statements for $V'_{\mathbf{i}}$ are obtained from those of $V_{\mathbf{i}}$ using the involution ψ^*. Hence, we only prove the theorem for $V_{\mathbf{i}}$.

Let V be an integrable U-module, $f \in V^*$ a weight vector with $\mathrm{wt}\, f = \mu$ and $v \in V$. Suppose that the right action of E_i and F_i on f is nilpotent for all $i \in I$. By Corollary 3.7, there exist $\lambda' \in P_+$ and a right U-module homomorphism $\varrho' : V(\lambda')^\star \otimes V(w^{-1}\mu - \lambda')^\star \to V^*$ such that $\varrho'(f_{w\lambda'} \otimes f_{w(w^{-1}\mu-\lambda')}) = f$. Then, by the argument similar to the beginning of the proof of Proposition 3.10,

$$c^V_{f,v} = \sum_{b\in B(\lambda'), b'\in B(w^{-1}\mu-\lambda')} a_{b,b'} c^{\lambda'}_{f_{w\lambda'},g_b} c^{w^{-1}\mu-\lambda'}_{f_{w(w^{-1}\mu-\lambda')},g_{b'}} \text{ for some } a_{b,b'} \in \mathbb{Q}(q). \tag{5.1}$$

This summation is well-defined in U^*.

We may assume that $-\lambda'' := w^{-1}\mu - \lambda' \in -P_+$. Then,

$$\begin{aligned}
&(\pi_{\mathbf{i}}^-)(c^{-\lambda''}_{f_{-w\lambda''},g_{b'}}).|0\rangle_{\mathbf{i}} \\
&= \sum_{b_1,\dots,b_{l-1}\in B(-\lambda'')} \iota^*_{i_1}(c^{-\lambda''}_{f_{-w\lambda''},g_{b_1}}).|0\rangle_{i_1} \otimes \cdots \otimes \iota^*_{i_k}(c^{-\lambda''}_{f_{b_{k-1}},g_{b_k}}).|0\rangle_{i_k} \otimes \cdots \otimes \iota^*_{i_l}(c^{-\lambda''}_{f_{b_{l-1}},g_{b'}}).|0\rangle_{i_l} \\
&= \iota^*_{i_1}(c^{-\lambda''}_{f_{-w\lambda''},v_{-s_{i_2}\cdots s_{i_l}\lambda''}}).|0\rangle_{i_1} \otimes \sum_{b_2,\dots,b_{l-1}\in B(-\lambda'')} \iota^*_{i_2}(c^{-\lambda''}_{f_{-s_{i_2}\cdots s_{i_l}\lambda''},g_{b_2}}).|0\rangle_{i_2} \otimes \cdots \otimes \iota^*_{i_l}(c^{-\lambda''}_{f_{b_{l-1}},g_{b'}}).|0\rangle_{i_l} \\
&\cdots \\
&= \delta_{g_{b'},v_{-\lambda''}} \iota^*_{i_1}(c^{-\lambda''}_{f_{-w\lambda''},v_{-s_{i_2}\cdots s_{i_l}\lambda''}}).|0\rangle_{i_1} \otimes \cdots \otimes \iota^*_{i_k}(c^{-\lambda''}_{f_{-s_{i_k}\cdots s_{i_l}\lambda''},v_{-s_{i_{k+1}}\cdots s_{i_l}\lambda''}}).|0\rangle_{i_k} \\
&\qquad\qquad \otimes \cdots \otimes \iota^*_{i_l}(c^{-\lambda''}_{f_{-s_{i_l}\lambda''},v_{-\lambda''}}).|0\rangle_{i_l} \\
&= \delta_{g_{b'},v_{-\lambda''}} (\prod_{k=1}^{l} (-q_{i_k})^{\langle s_{i_{k+1}}\cdots s_{i_l}\lambda'', \alpha^\vee_{i_k}\rangle})|0\rangle_{\mathbf{i}} (=: D_{\lambda'',w}|0\rangle_{\mathbf{i}}),
\end{aligned} \tag{5.2}$$

here the second equality follows from Lemma 3.8 and the fact that the elements of $-s_{i_1}w\lambda'' - \mathbb{Z}_{>0}\alpha_{i_1}$ are not weights of $V(-\lambda'')$, and the last equality follows from $\iota^*_{i_k}(c^{-\lambda''}_{f_{-s_{i_k}\cdots s_{i_l}\lambda''},v_{-s_{i_{k+1}}\cdots s_{i_l}\lambda''}}) = c_{12}^{\langle -s_{i_k}\cdots s_{i_l}\lambda'', \alpha^\vee_{i_k}\rangle}$. See Definition 3.1 and 3.2. Note that $D_{\lambda'',w}$ does not depend on the choice of $\mathbf{i}$ because $\sum_{k=1}^{l}\langle s_{i_{k+1}}\cdots s_{i_l}\lambda'', \alpha^\vee_{i_k}\rangle = \langle \lambda'' - w\lambda'', \sum_{i\in I}\varpi_i^\vee\rangle$ and $\sum_{k=1}^{l}(s_{i_{k+1}}\cdots s_{i_l}\lambda'', \alpha_{i_k}) = (\lambda'' - w\lambda'', \sum_{i\in I}\varpi_i)$. Therefore, we have

$$((\pi_{i_1}\otimes\cdots\otimes\pi_{i_l})\circ\iota^*_{\mathbf{i}}\circ\hat{\Delta}_l)(c^V_{f,v}).|0\rangle_{\mathbf{i}} = D_{\lambda'',w}\sum_{b\in B(\lambda')} a_{b,v_{-\lambda''}}\pi^+_{\mathbf{i}}(c^{\lambda'}_{f_{w\lambda'},g_b}).|0\rangle_{\mathbf{i}}.$$

Note that the right-hand side is a finite sum.

Hence the proof of the theorem is completed by showing that the $\mathbb{Q}(q)$-linear isomorphism $\Psi^{+,\mathbf{i}}_{\mathrm{KOY}} : V_{\mathbf{i}} \to U^+/U^+(w)^\perp, |\mathbf{c}\rangle_{\mathbf{i}} \to [E^{\mathbf{c}}_{\mathbf{i}}]_w$ satisfies

$$\Psi^{+,\mathbf{i}}_{\mathrm{KOY}}(\pi^+_{\mathbf{i}}(C).|0\rangle_{\mathbf{i}}) = h^w_+(C).[1]_w \text{ for all } C \in {}^{w(\mathrm{hi})}A_q[\mathfrak{g}],$$

(See Corollary 4.14.) where the $\check{U}^{\geq 0}$-module structure on $U^+/U^+(w)^\perp$ is given by

$$E_i.[u]_w = [E_i u]_w \text{ and } K_\lambda.[1]_w = [1]_w \text{ for } i \in I \text{ and } \lambda \in P.$$

Note that $U^+(w)^\perp$ is the left ideal of U^+ by Corollary 5.7. Therefore, it suffices to show that

(a) $(h_+^w)^{-1}(K_\lambda).|0\rangle_{\mathbf{i}}(= c^\lambda_{f_{w\lambda},v_\lambda}.|0\rangle_{\mathbf{i}}) = |0\rangle_{\mathbf{i}}$ for all $\lambda \in P_+$; therefore, the space $\pi^+_{\mathbf{i}}({}^{w(\mathrm{hi})}A_q[\mathfrak{g}]).|0\rangle_{\mathbf{i}}$ has the ${}^{w(\mathrm{hi})}A_q[\mathfrak{g}]_{w}{}_{S}$-module structure.

(b) $(h_+^w)^{-1}(G^+(\tilde{b}^\omega)).|0\rangle_{\mathbf{i}} = \sum_{\mathbf{c}\in\mathbb{Z}^l_{\geq 0}} {}_{\mathbf{i}}\zeta^{\tilde{b}}_{\mathbf{c}}|\mathbf{c}\rangle_{\mathbf{i}}$ for all $\tilde{b} \in B(\infty)$ (See Proposition 5.9 and Remark 5.10).

The statement (a) follows from the calculation similar to (5.2).

By the way, $\tilde{\Omega}(c^\lambda_{f_\lambda,G^-(\tilde{b}).v_\lambda}) = K_{-\lambda} \otimes h_+^w(c^\lambda_{f_{w\lambda},G^-(\tilde{b}).v_\lambda})$. See Proposition 4.11 and Corollary 4.14. Hence, by Proposition 4.17, there exist unique polynomials $\{f_{\tilde{b},\tilde{b}'}(x)\}_{\tilde{b},\tilde{b}'\in B(\infty)}$ in $x = (x_i)_{i\in I}$ with coefficients $\mathbb{Z}[q^{\pm 1}]$ such that

$$h_+^w(c^\lambda_{f_{w\lambda},G^-(\tilde{b}).v_\lambda}) = \sum\nolimits_{\tilde{b}'\in B(\infty)} f_{\tilde{b},\tilde{b}'}(q_i^{2\langle\lambda,\alpha_i^\vee\rangle})G^+((\tilde{b}')^\omega)K_\lambda,$$

where $f_{\tilde{b},\tilde{b}'}(q_i^{2\langle\lambda,\alpha_i^\vee\rangle})$ denotes the element of $\mathbb{Z}[q^{\pm 1}]$ obtained by substituting $q_i^{2\langle\lambda,\alpha_i^\vee\rangle}$ for x_i ($i \in I$) in $f_{\tilde{b},\tilde{b}'}(x)$.

By Proposition 4.17, the polynomials $\{f_{\tilde{b},\tilde{b}'}(x)\}_{\tilde{b},\tilde{b}'\in B(\infty)}$ satisfy the properties $f_{\tilde{b},\tilde{b}'}(0) = \delta_{\tilde{b},\tilde{b}'}$ and $f_{\tilde{b},\tilde{b}'}(x) = 0$ unless $\mathrm{wt}\,\tilde{b} = \mathrm{wt}\,\tilde{b}'$. In particular, for any $\tilde{b} \in B(\infty)$, the number of $\tilde{b}'$ with $f_{\tilde{b},\tilde{b}'}(x) \neq 0$ is finite. Therefore, the calculation results of $\pi^+_{\mathbf{i}}(c^\lambda_{f_{w\lambda},G^-(\tilde{b}).v_\lambda}).|0\rangle_{\mathbf{i}}$ for various $\lambda \in P_+$ are uniformly described by using the polynomials $\{f_{\tilde{b},\tilde{b}'}(x)\}_{\tilde{b}'\in B(\infty)}$ and the constant term in this description corresponds to $(h_+^w)^{-1}(G^+(\tilde{b})).|0\rangle_{\mathbf{i}}$. Hence, the statement (b) follows from Theorem 5.20. □

In the proof of Corollary 5.21, we showed the following corollary. This is Kuniba-Okado-Yamada and Tanisaki's common structure theorem, which has originally been proved in [13] (Finite and the longest element of W case) and [26] (Symmetrizable Kac-Moody and arbitrary $w \in W$ case). Incidentally, our formulation and convention are slightly different from those of original papers. See also Remark 5.24.

Corollary 5.23. *Let $w \in W$ and $\mathbf{i} = (i_1, \dots, i_l) \in I(w)$. We define the $\check{U}^{\geq 0}$-module (resp. $\check{U}^{\leq 0}$-module) structure on $U^+/U^+(w)^\perp$ (resp. $U^-/U^-(w)^\perp$) by*

$$X_i.[u]_w = [X_i u]_w \text{ for } i \in I,\ X = E \text{ (resp. } F) \text{ and } K_\lambda.[1]_w = [1]_w \text{ for } \lambda \in P.$$

Note that $U^\pm(w)^\perp$ is a left ideal of $U^\pm$ by Corollary 5.7.

Define the $\mathbb{Q}(q)$-linear isomorphisms $\Psi^{\pm,w}_{\mathrm{KOY}}$ by

$$\begin{aligned} \Psi^{+,w}_{\mathrm{KOY}} &: V_w \to U^+/U^+(w)^\perp, |\mathbf{c}\rangle_{\mathbf{i}} \to [E^{\mathbf{c}}_{\mathbf{i}}]_w \\ \Psi^{-,w}_{\mathrm{KOY}} &: V'_w \to U^-/U^-(w^{-1})^\perp, \langle\mathbf{c}|_{\mathbf{i}} \mapsto [F^{\overline{\mathbf{c}}}_{\overline{\mathbf{i}}}]_{w^{-1}}. \end{aligned}$$

Recall Notation 5.16. *Then, for $C \in {}^{w(\mathrm{hi})}A_q[\mathfrak{g}]$, $C' \in A_q[\mathfrak{g}]^{w^{-1}(\mathrm{hi})}$ and $\Lambda \in V_w$, $\Xi \in V'_w$, we have*

$$\Psi^{+,w}_{\mathrm{KOY}}(\pi^+_{\mathbf{i}}(C).\Lambda) = h_+^w(C).\Psi^{+,w}_{\mathrm{KOY}}(\Lambda) \text{ and } \Psi^{-,w}_{\mathrm{KOY}}(\pi'^+_{\mathbf{i}}(C').\Xi) = h_-^w(C').\Psi^{-,w}_{\mathrm{KOY}}(\Xi).$$

In particular, the maps $\Psi^{\pm,w}_{\mathrm{KOY}}$ do not depend on the choice of $\mathbf{i} \in I(w)$.

Moreover, for an integrable U-module V and weight vectors $v \in V$, $f \in V^*$ such that the right action of E_i and F_i on f is nilpotent for all $i \in I$, we have

- $((\pi_{i_1}\otimes\cdots\otimes\pi_{i_l})\circ\iota_{\mathbf{i}}^*\circ\hat{\Delta}_l)(c^V_{f,v})$ *acts on* $U^+/U^+(w)^\perp$ *via* $\Psi^{+,w}_{\mathrm{KOY}}$ *as an operator of degree* $w^{-1}\,\mathrm{wt}\,f-\mathrm{wt}\,v$*, and*
- $((\pi'_{i_1}\otimes\cdots\otimes\pi'_{i_l})\circ\iota_{\mathbf{i}}^*\circ\hat{\Delta}_l)(c^V_{f,v})$ *acts on* $U^-/U^-(w^{-1})^\perp$ *via* $\Psi^{-,w}_{\mathrm{KOY}}$ *as an operator of degree* $\mathrm{wt}\,f-w\,\mathrm{wt}\,v$.

Remark 5.24. In this setting, for $i\in I$,

$$h^w_+((1-q_i^2)^{-1}c^{\varpi_i}_{f_{w\varpi_i},F_i.v_{\varpi_i}}(c^{\varpi_i}_{f_{w\varpi_i},v_{\varpi_i}})^{-1})=E_i,\ \text{and}$$
$$h^w_-((1-q_i^2)^{-1}c^{\varpi_i}_{f_{\varpi_i}.E_i,v_{w^{-1}\varpi_i}}(c^{\varpi_i}_{f_{\varpi_i},v_{w^{-1}\varpi_i}})^{-1})=F_i.$$

Hence this corollary includes the statement of the Kuniba-Okado-Yamada conjecture [13, Conjecture 1], which has originally been proved by Saito [23, Corollary 4.3.3] and Tanisaki [26, Proposition 7.6] independently.

Definition 5.25. For $w\in W$, we have the decomposition into $\mathbb{Q}(q)$-subspaces

$$V_w=\bigoplus_{\alpha\in Q_+}(\Psi^{+,w}_{\mathrm{KOY}})^{-1}((U^+/U^+(w)^\perp)_\alpha)(=:\bigoplus_{\alpha\in Q_+}(V_w)_\alpha),$$
$$V'_w=\bigoplus_{\alpha\in Q_+}(\Psi^{-,w}_{\mathrm{KOY}})^{-1}((U^-/U^-(w^{-1})^\perp)_{-\alpha})(=:\bigoplus_{\alpha\in Q_+}(V'_w)_{-\alpha}).$$

These decompositions are said to be the weight space decomposition. The weight spaces of V_w and V'_w are simultaneous eigenspaces corresponding to the actions of the elements ${}_w\mathcal{S}$ and $\mathcal{S}_{w^{-1}}$ respectively.

Corollary 5.26. *Let* $w\in W$. *Then the isomorphism class of* $V_{\mathbf{i}}$ (*resp.* $V'_{\mathbf{i}}$) *as an* $A_q[\mathfrak{g}]^-$*-module does not depend on the choice of* $\mathbf{i}\in I(w)$. *Moreover, the* $A_q[\mathfrak{g}]^-$*-module* $V_{\mathbf{i}}$ (*resp.* $V'_{\mathbf{i}}$) *is generated by* $|0\rangle_{\mathbf{i}}$ (*resp.* $\langle 0|_{\mathbf{i}}$), *and, for any* $\mathbf{i},\mathbf{j}\in I(w)$, *an isomorphism* $V_{\mathbf{i}}\to V_{\mathbf{j}}$ (*resp.* $V'_{\mathbf{i}}\to V'_{\mathbf{j}}$) *of* $A_q[\mathfrak{g}]^-$*-modules is given by* $|0\rangle_{\mathbf{i}}\mapsto|0\rangle_{\mathbf{j}}$ (*resp.* $\langle 0|_{\mathbf{i}}\mapsto\langle 0|_{\mathbf{j}}$).

Definition 5.27. We will identify the $A_q[\mathfrak{g}]^-$-modules $V_{\mathbf{i}}$ (resp. $V'_{\mathbf{i}}$) ($\mathbf{i}\in I(w)$) via the isomorphisms in 5.26 and denote by π^-_w (resp. π'^-_w) the corresponding $\mathbb{Q}(q)$-algebra homomorphism $A_q[\mathfrak{g}]^-\to\mathrm{End}_{\mathbb{Q}(q)}(V_w)$(resp. $\mathrm{End}_{\mathbb{Q}(q)}(V'_w)$).

Proof of Corollary 5.26. The statements for $V'_{\mathbf{i}}$ are obtained from those of $V_{\mathbf{i}}$ using the involution ψ^*. Hence, we only prove the theorem for $V_{\mathbf{i}}$.

For an integrable U-module V and weight vectors $v\in V, f\in V^*$, there exist $\lambda,\lambda'\in P_+$ and a left U-module homomorphism $\varrho':V(-\lambda)\otimes V(\lambda')\to V$ such that $\varrho'(v_{-\lambda}\otimes v_{\lambda'})=v$ by Proposition 3.4. Then, by the argument similar to the beginning of the proof of Proposition 3.10, we have

$$c^V_{f,v}=\sum_{b\in B(-\lambda),b'\in B(\lambda')}a_{b,b'}c^{-\lambda}_{f_b,v_{-\lambda}}c^{\lambda'}_{f_{b'},v_{\lambda'}}\ \text{for some}\ a_{b,b'}\in\mathbb{Q}(q).$$

This summation is well-defined in U^*. Hence,

$$(\iota_{\mathbf{i}}^*\circ\hat{\Delta}_l)(c^V_{f,v})=\sum_{b\in B(-\lambda),b'\in B(\lambda')}a_{b,b'}(\iota_{\mathbf{i}}^*\circ\hat{\Delta}_l)(c^{-\lambda}_{f_b,v_{-\lambda}})(\iota_{\mathbf{i}}^*\circ\hat{\Delta}_l)(c^{\lambda'}_{f_{b'},v_{\lambda'}}).$$

The right-hand side is, in fact, a finite sum. By the calculation similar to (5.2), we obtain

$$((\pi_{i_1}\otimes\cdots\otimes\pi_{i_l})\circ\iota_{\mathbf{i}}^*\circ\hat{\Delta}_l)(c^V_{f,v}).|0\rangle_{\mathbf{i}}=\sum_{b\in B(-\lambda)}a_{b,v_{w\lambda'}}\pi^-_{\mathbf{i}}(c^{-\lambda}_{f_b,v_{-\lambda}}).|0\rangle_{\mathbf{i}}.$$

On the other hand, by Corollary 5.21, any element of $V_{\mathbf{i}}$ is of the form $\pi_{\mathbf{i}}^{+}(C).|0\rangle_{\mathbf{i}}$ ($C \in A_q[\mathfrak{g}]^+$). Therefore, the $A_q[\mathfrak{g}]^-$-module $V_{\mathbf{i}}$ is generated by $|0\rangle_{\mathbf{i}}$. The other statements follows from Corollary 5.21 and the argument in the first half of the proof of Corollary 5.21. □

Before ending this subsection, we mention a bilinear from on V_w and prove the irreducibility of V_w.

Definition 5.28. Let $w \in W$ and $\mathbf{i} = (i_1, \ldots, i_l) \in I(w)$. Then we define the $\mathbb{Q}(q)$-bilinear form $(\ ,\ )_w : V_w \times V_w \to \mathbb{Q}(q)$ by

$$(|\mathbf{m}\rangle_{\mathbf{i}}, |\mathbf{m}'\rangle_{\mathbf{i}})_w = (|m_1\rangle_{i_1}, |m_1'\rangle_{i_1})_{i_1} \cdots (|m_l\rangle_{i_l}, |m_l'\rangle_{i_l})_{i_l} = \delta_{\mathbf{m},\mathbf{m}'} \prod_{s=1}^{l} \prod_{t=1}^{m_s} (1 - q_{i_s}^{2t})^{-1}.$$

Then this form satisfies $(\pi_w^+(C).\Lambda, \Lambda')_w = (\Lambda, \pi_w^-((\psi \circ S)^*(C)).\Lambda')_w$ for $C \in A_q[\mathfrak{g}]^+$, $\Lambda, \Lambda' \in V_w'$, and does not depend on the choice of $\mathbf{i}$. Note that $\psi \circ S$ is a $\mathbb{Q}(q)$-algebra involution of U.

Corollary 5.29. *Let $w \in W$. Define a $\mathbb{Q}(q)$-bilinear form $(\ ,\ )_w$ on $U^+/U^+(w)^\perp$ by $([X]_w, [X']_w)_w := (p_w(X), p_w(X'))_{\mathrm{pos}}$ where $p_w : U^+ = U^+(w) \oplus U^+(w)^\perp \to U^+(w)$ is the projection. Then,*

$$(\Lambda, \Lambda')_w = (\Psi_{\mathrm{KOY}}^{+,w}(\Lambda), \Psi_{\mathrm{KOY}}^{+,w}(\Lambda'))_w \text{ for all } \Lambda, \Lambda' \in V_w.$$

Proof. This statement follows from the direct calculation of the form $(\ ,\ )_w$ on $U^+/U^+(w)^\perp$ for PBW-type elements. (See Proposition 5.5.) □

Proposition 5.30. *For $w \in W$, the $A_q[\mathfrak{g}]^+$-modules V_w and V_w' are irreducible.*

Proof. Suppose that there exists $A_q[\mathfrak{g}]^+$-submodule V' such that $0 \subsetneq V' \subsetneq V_w$. Then $V' = \bigoplus_{\alpha \in Q_+} (V' \cap (V_w)_\alpha)$. Note that $V' \cap (V_w)_0 = 0$ because the $A_q[\mathfrak{g}]^+$-module V_w is generated by $|0\rangle_w$. Therefore $(|0\rangle_w, V')_w = 0$.

Let $0 \neq \Lambda' \in V'$. Since the form $(\ ,\ )_w$ is nondegenerate, there exists $\Lambda \in V_w$ such that $(\Lambda, \Lambda')_w \neq 0$. By Corollary 5.26, there exists $C^- \in A_q[\mathfrak{g}]^-$ such that $\pi_w^-(C^-).|0\rangle_w = \Lambda$. Then $(\Lambda, \Lambda')_w = (|0\rangle_w, \pi_w^+((\psi \circ S)^*(C^-)).\Lambda')_w = 0$, which contradicts $(\Lambda, \Lambda')_w \neq 0$. □

As in the proof of Proposition 5.30, we also obtain the following proposition.

Proposition 5.31. *For $w \in W$, the $A_q[\mathfrak{g}]^-$-modules V_w and V_w' are irreducible.*

5.3. The reducible $A_q[\mathfrak{g}]^+$-modules $\tilde{V}_w$. The $A_q[\mathfrak{g}]^+$-module $\tilde{V}_w := V_{w^{-1}}' \otimes V_w$ is reducible but has a reasonable structure, which is compatible with the embedding $\tilde{\Omega}$.

Theorem 5.32. *Let $w \in W$. As in* Corollary 5.23*, we regard $U^-/U^-(w)^\perp \otimes U^+/U^+(w)^\perp$ as a $\check{U}^{\leq 0} \otimes \check{U}^{\geq 0}$-module and, via $\tilde{\Omega}$, an $A_q[\mathfrak{g}]_S^+$-module. See* Proposition 4.11.

Then the $A_q[\mathfrak{g}]^+$-module structure on $\tilde{V}_w := V_{w^{-1}}' \otimes V_w$ can be extended to the $A_q[\mathfrak{g}]_S^+$-module structure and, as an $A_q[\mathfrak{g}]_S^+$-module, $\tilde{V}_w$ is isomorphic to $U^-/U^-(w)^\perp \otimes U^+/U^+(w)^\perp$ where a corresponding isomorphism $\tilde{\Psi}_{\mathrm{KOY}}^w$ is given by $\langle 0|_{w^{-1}} \otimes |0\rangle_w \mapsto [1]_w \otimes [1]_w$.

In particular, the $A_q[\mathfrak{g}]^+$-module $\tilde{V}_w$ is generated by $\langle 0|_{w^{-1}} \otimes |0\rangle_w$ and decomposed into the finite dimensional eigenspaces of the actions of S, and any two linearly independent *eigenvectors generate different infinite dimensional submodules.*

Moreover, for $\mathbf{i}, \mathbf{i}' \in I(w)$ *(length* l*) and* $\mathbf{c}, \mathbf{c}' \in \mathbb{Z}^{l}_{\geq 0}$,

$$\tilde{\Psi}^{w}_{\mathrm{KOY}}(\langle \overline{\mathbf{c}}|_{\bar{\mathbf{i}}} \otimes |\mathbf{c}'\rangle_{\mathbf{i}'}) = [F^{\mathbf{c}}_{\mathbf{i}}]_{w} \otimes [E^{\mathbf{c}'}_{\mathbf{i}'}]_{w} + \sum_{\substack{Y\in U^{-}, X\in U^{+} \text{ homogeneous} \\ \mathrm{wt}\, Y > \mathrm{wt}\, F^{\mathbf{c}}_{\mathbf{i}}, \mathrm{wt}\, X < \mathrm{wt}\, E^{\mathbf{c}'}_{\mathbf{i}'}}} [Y]_{w} \otimes [X]_{w}. \tag{5.3}$$

REMARK-QUESTION 5.33. The summation in the right hand side of (5.3) is nontrivial, that is, $\tilde{\Psi}^{w}_{\mathrm{KOY}} \neq \Psi^{-,w^{-1}}_{\mathrm{KOY}} \otimes \Psi^{+,w}_{\mathrm{KOY}}$. Hence we have the nontrivial $\mathbb{Q}(q)$-linear automorphism

$$I^{w} := (\Psi^{-,w^{-1}}_{\mathrm{KOY}} \otimes \Psi^{+,w}_{\mathrm{KOY}}) \circ (\tilde{\Psi}^{w}_{\mathrm{KOY}})^{-1}$$

on $U^{-}/U^{-}(w)^{\perp} \otimes U^{+}/U^{+}(w)^{\perp}$. We do not know whether the map I^{w} is obtained without using representations of the quantized coordinate algebras, and has any significance in the structure theory of the quantized enveloping algebras. (See also Subsection 6.1.)

Proof of Theorem 5.32. For $\lambda \in P_{+}$, $c^{\lambda}_{f_{\lambda}, v_{\lambda}}.(\langle 0|_{w^{-1}} \otimes |0\rangle_{w}) = \langle 0|_{w^{-1}} \otimes |0\rangle_{w}$ by the calculation similar to (5.2). On the other hand, $\tilde{\Omega}(c^{\lambda}_{f_{\lambda}, v_{\lambda}}) = K_{-\lambda} \otimes K_{\lambda}$. Hence the actions of $c^{\lambda}_{f_{\lambda}, v_{\lambda}}$'s for $\lambda \in P_{+}$ on $(A_{q}[\mathfrak{g}]^{e(\mathrm{hi})})({}^{e(\mathrm{hi})}A_{q}[\mathfrak{g}]).(\langle 0|_{w^{-1}} \otimes |0\rangle_{w})$ is invertible and, by Corollary 4.13, this subspace has a $A_{q}[\mathfrak{g}]^{+}_{\mathcal{S}}$-module structure.

Now, for homogeneous elements $N^{\pm} \in U^{\pm}(w)^{\perp}$ and $X^{\pm} \in U^{\pm}$,

$$\tilde{\Omega}^{-1}(N^{-} \otimes X^{+}) = c^{\lambda}_{f_{\lambda}, v} c^{\lambda'}_{f, v_{\lambda'}} (c^{\lambda''}_{f_{\lambda''}, v_{\lambda''}})^{-1} \text{ and } \tilde{\Omega}^{-1}(X^{-} \otimes N^{+}) = c^{\mu}_{f', v_{\mu}} c^{\mu'}_{f_{\mu'}, v'} (c^{\mu''}_{f_{\mu''}, v_{\mu''}})^{-1}$$

for some $\lambda, \lambda', \lambda'', \mu, \mu', \mu'' \in P_{+}$, $v \in V(\lambda)_{\lambda - \mathrm{wt}\, X^{+}}$, $f \in V(\lambda')^{\star}_{\lambda' + \mathrm{wt}\, N^{-}}$, $f' \in V(\mu)^{\star}_{\mu + \mathrm{wt}\, X^{-}}$ and $v' \in V(\mu')_{\mu' - \mathrm{wt}\, N^{+}}$ with $\tilde{\Omega}(c^{\lambda'}_{f, v_{\lambda'}}) = N^{-}K_{-\lambda'} \otimes K_{\lambda'}$ and $\tilde{\Omega}(c^{\mu'}_{f_{\mu'}, v}) = K_{-\mu'} \otimes N^{+}K_{\mu'}$. Then, by Corollary 5.23,

$$c^{\lambda'}_{f, v_{\lambda'}}.(\langle 0|_{w^{-1}} \otimes |0\rangle_{w}) = (c^{\lambda'}_{f, v_{w\lambda'}}.\langle 0|_{w^{-1}}) \otimes |0\rangle_{w} = 0.$$

Similarly, $c^{\mu'}_{f_{\mu'}, v'}.(\langle 0|_{w^{-1}} \otimes |0\rangle_{w}) = 0$. Therefore,

$$\tilde{\Omega}^{-1}(N^{-} \otimes X^{+}).(\langle 0|_{w^{-1}} \otimes |0\rangle_{w}) = \tilde{\Omega}^{-1}(X^{-} \otimes N^{+}).(\langle 0|_{w^{-1}} \otimes |0\rangle_{w}) = 0.$$

Hence, the map $\bigoplus_{\lambda \in P} U^{-}K_{-\lambda} \otimes U^{+}K_{\lambda} \to A_{q}[\mathfrak{g}]^{+}_{\mathcal{S}}.(\langle 0|_{w^{-1}} \otimes |0\rangle_{w})$ given by $\tilde{X} \mapsto \tilde{\Omega}^{-1}(\tilde{X}).(\langle 0|_{w^{-1}} \otimes |0\rangle_{w})$ (cf. Proposition 4.11) factors through $\bigoplus_{\lambda \in P} U^{-}K_{-\lambda} \otimes U^{+}K_{\lambda} \to U^{-}/U^{-}(w)^{\perp} \otimes U^{+}/U^{+}(w)^{\perp}$, $\tilde{X} \mapsto \tilde{X}.([1]_{w} \otimes [1]_{w})$. Let us denote by $\tilde{\Phi}$ the induced homomorphism $U^{-}/U^{-}(w)^{\perp} \otimes U^{+}/U^{+}(w)^{\perp} \to A_{q}[\mathfrak{g}]^{+}_{\mathcal{S}}.(\langle 0|_{w^{-1}} \otimes |0\rangle_{w})$.

What is left is to show the equality

$$\tilde{\Phi}([F^{\mathbf{c}}_{\mathbf{i}}]_{w} \otimes [E^{\mathbf{c}'}_{\mathbf{i}'}]_{w}) = \langle \overline{\mathbf{c}}|_{\bar{\mathbf{i}}} \otimes |\mathbf{c}'\rangle_{\mathbf{i}'} + \sum_{\substack{Y\in U^{-}, X\in U^{+} \text{ homogeneous} \\ \mathrm{wt}\, Y > \mathrm{wt}\, F^{\mathbf{c}}_{\mathbf{i}}, \mathrm{wt}\, X < \mathrm{wt}\, E^{\mathbf{c}'}_{\mathbf{i}'}}} (\Psi^{-,w^{-1}}_{\mathrm{KOY}})^{-1}([Y]_{w}) \otimes (\Psi^{+,w}_{\mathrm{KOY}})^{-1}([X]_{w}),$$

for any $\mathbf{c}, \mathbf{c}'$ because this implies the injectivity of $\tilde{\Phi}$, $A_{q}[\mathfrak{g}]^{+}_{\mathcal{S}}.(\langle 0|_{w^{-1}} \otimes |0\rangle_{w}) = \tilde{V}_{w}$ and the equality (5.3). We have

$$\begin{aligned}\tilde{\Phi}([F^{\mathbf{c}}_{\mathbf{i}}]_{w} \otimes [E^{\mathbf{c}'}_{\mathbf{i}'}]_{w}) &= \tilde{\Phi}(\tilde{\Omega}^{-1}(F^{\mathbf{c}}_{\mathbf{i}} \otimes E^{\mathbf{c}'}_{\mathbf{i}'}).([1]_{w} \otimes [1]_{w})) \\ &= \tilde{\Omega}^{-1}(F^{\mathbf{c}}_{\mathbf{i}} \otimes E^{\mathbf{c}'}_{\mathbf{i}'}).(\langle 0|_{w^{-1}} \otimes |0\rangle_{w}).\end{aligned}$$

There exist $\lambda, \lambda' \in P_+$, $v \in V(\lambda)_{\lambda - \mathrm{wt}\, E^{\mathbf{c}'}_{\mathbf{i}'}}$ and $f \in V(\lambda')^{\star}_{\lambda' + \mathrm{wt}\, F^{\mathbf{c}}_{\mathbf{i}}}$ such that $\tilde{\Omega}(c^{\lambda}_{f_\lambda, v}) = K_{-\lambda} \otimes E^{\mathbf{c}'}_{\mathbf{i}'} K_\lambda$, $\tilde{\Omega}(c^{\lambda'}_{f, v_{\lambda'}}) = F^{\mathbf{c}}_{\mathbf{i}} K_{-\lambda'} \otimes K_{\lambda'}$. Then, $\tilde{\Omega}^{-1}(F^{\mathbf{c}}_{\mathbf{i}} \otimes E^{\mathbf{c}'}_{\mathbf{i}'}) = q^{(\lambda, \mathrm{wt}\, F^{\mathbf{c}}_{\mathbf{i}})} c^{\lambda}_{f_\lambda, v} c^{\lambda'}_{f, v_{\lambda'}} (c^{\lambda+\lambda'}_{f_{\lambda+\lambda'}, v_{\lambda+\lambda'}})^{-1}$. Hence, by Corollary 5.23,

$$
\begin{aligned}
&\tilde{\Omega}^{-1}(F^{\mathbf{c}}_{\mathbf{i}} \otimes E^{\mathbf{c}'}_{\mathbf{i}'}).(\langle 0|_{\bar{\mathbf{i}}} \otimes |0\rangle_{\mathbf{i}'})\\
&= q^{(\lambda, \mathrm{wt}\, F^{\mathbf{c}}_{\mathbf{i}})} c^{\lambda}_{f_\lambda, v}.(\langle \overline{\mathbf{c}}|_{\bar{\mathbf{i}}} \otimes |0\rangle_{\mathbf{i}'})\\
&= q^{(\lambda, \mathrm{wt}\, F^{\mathbf{c}}_{\mathbf{i}})} \left((c^{\lambda}_{f_\lambda, v_{w\lambda}}.\langle \overline{\mathbf{c}}|_{\bar{\mathbf{i}}}) \otimes (c^{\lambda}_{f_{w\lambda}, v}.|0\rangle_{\mathbf{i}'}) + \sum_{b \in B(\lambda), \mathrm{wt}\, b \neq w\lambda} (c^{\lambda}_{f_\lambda, g_b}.\langle \overline{\mathbf{c}}|_{\bar{\mathbf{i}}}) \otimes (c^{\lambda}_{f_b, v}.|0\rangle_{\mathbf{i}'}) \right)\\
&= \langle \overline{\mathbf{c}}|_{\bar{\mathbf{i}}} \otimes |\mathbf{c}'\rangle_{\mathbf{i}'} + \sum_{\substack{Y \in U^-, X \in U^+ \text{ homogeneous}\\ \mathrm{wt}\, X' < \mathrm{wt}\, E^{\mathbf{c}'}_{\mathbf{i}'}}} (\Psi^{-, w^{-1}}_{\mathrm{KOY}})^{-1}([Y]_w) \otimes (\Psi^{+, w}_{\mathrm{KOY}})^{-1}([X]_w).
\end{aligned}
$$

The last equality follows from Corollary 5.23 and the inequality $w^{-1}(\mathrm{wt}\, b) < \lambda$ for all $b \in B(\lambda)$ with $\mathrm{wt}\, b \neq w\lambda$.

On the other hand, $\tilde{\Omega}^{-1}(F^{\mathbf{c}}_{\mathbf{i}} \otimes E^{\mathbf{c}'}_{\mathbf{i}'}) = q^{-(\lambda', \mathrm{wt}\, E^{\mathbf{c}'}_{\mathbf{i}'})} c^{\lambda'}_{f, v_{\lambda'}} c^{\lambda}_{f_\lambda, v} (c^{\lambda+\lambda'}_{f_{\lambda+\lambda'}, v_{\lambda+\lambda'}})^{-1}$. Hence the similar argument shows that

$$
\begin{aligned}
&\tilde{\Omega}^{-1}(F^{\mathbf{c}}_{\mathbf{i}} \otimes E^{\mathbf{c}'}_{\mathbf{i}'}).(\langle 0|_{\bar{\mathbf{i}}} \otimes |0\rangle_{\mathbf{i}'})\\
&= \langle \overline{\mathbf{c}}|_{\bar{\mathbf{i}}} \otimes |\mathbf{c}'\rangle_{\mathbf{i}'} + \sum_{\substack{Y \in U^-, X \in U^+ \text{ homogeneous}\\ \mathrm{wt}\, Y > \mathrm{wt}\, F^{\mathbf{c}}_{\mathbf{i}}}} (\Psi^{-, w^{-1}}_{\mathrm{KOY}})^{-1}([Y]_w) \otimes (\Psi^{+, w}_{\mathrm{KOY}})^{-1}([X]_w).
\end{aligned}
$$

These equalities completes the proof. □

Corollary 5.34. *For $w, w' \in W$ with $w \leq_r w'$ ($\leq_r$ is the weak right Bruhat order), the $A_q[\mathfrak{g}]^+$-modules $V'_{w^{-1}} \otimes V_{w'}$ (resp. $V'_{w'^{-1}} \otimes V_w$) are generated by $\langle 0|_{w^{-1}} \otimes |0\rangle_{w'}$ (resp. $\langle 0|_{w'^{-1}} \otimes |0\rangle_w$).*

Let $w \in W$. By Remark 5.17 and Theorem 5.32, we have the $A_q[\mathfrak{g}]^+$-module homomorphism $\Gamma_w : (V_w)^{\psi^*} \otimes V_w (= \tilde{V}_w) \to \mathbb{Q}(q) (= V_e)$ given by $|0\rangle_w \otimes |0\rangle_w \mapsto 1 (= |0\rangle_\emptyset)$.

Corollary 5.35. *Let $w \in W$. For $\Lambda, \Lambda' \in V_w$, we have $\Gamma_w(\Lambda \otimes \Lambda') = (\Lambda, \Lambda')_w$.*

Proof. Let $\lambda \in P_+$ and $v, v' \in V(\lambda)$. Then,

$$
c^{\lambda}_{v^*, v'} \otimes \varepsilon = \sum\nolimits_{b_1, b_2 \in B(\lambda)} (c^{\lambda}_{v^*, g_{b_1}} \otimes c^{\lambda}_{f_{b_1}, g_{b_2}})(\varepsilon \otimes S^*(c^{\lambda}_{f_{b_2}, v'}))
$$

in $(U \otimes U)^*$, where $S^* : U^* \to U^*$ is the $\mathbb{Q}(q)$-algebra anti-involution given by $F \mapsto F \circ S$. Then, for $\Lambda, \Lambda' \in V_w$,

$$
\begin{aligned}
(5.4) \qquad &\Gamma_w((\pi^+_w(c^{\lambda}_{v'^*, v}).\Lambda) \otimes \Lambda')\\
&= \Gamma_w(((\pi^+_w \circ \psi^*)(c^{\lambda}_{v^*, v'}).\Lambda) \otimes \Lambda')\\
&= \sum_{b_1, b_2 \in B(\lambda)} \Gamma_w(((\pi^+_w \circ \psi^*)(c^{\lambda}_{v^*, g_{b_1}}) \otimes \pi^+_w(c^{\lambda}_{f_{b_1}, g_{b_2}}))(1 \otimes \pi^-_w(S^*(c^{\lambda}_{f_{b_2}, v'}))).(\Lambda \otimes \Lambda'))\\
&= \sum_{b \in B(\lambda)} \Gamma_w(((\pi^+_w \circ \psi^*) \otimes \pi^+_w)(c^{\lambda}_{v^*, g_b}).(\Lambda \otimes \pi^-_w(S^*(c^{\lambda}_{f_b, v'})).\Lambda'))\\
&= \sum_{b \in B(\lambda)} (v, g_b)_\lambda.\Gamma_w(\Lambda \otimes \pi^-_w(S^*(c^{\lambda}_{f_b, v'})).\Lambda')
\end{aligned}
$$

$$= \Gamma_w(\Lambda \otimes \pi_w^-((\psi \circ S)^*(c^\lambda_{v'^*,v})).\Lambda').$$

Moreover, for $\mathbf{i} \in I(w)$ and $(0,\dots,0) \neq \mathbf{m} \in \mathbb{Z}^l_{\geq 0}$, there exist $\lambda \in P_+$ and weight vectors $f \in V(\lambda)^\star$, $v \in V(\lambda)$ with $\mathrm{wt}\, f \neq \mathrm{wt}\, v$ such that $((\pi_w^+ \circ \psi^*) \otimes \pi_w^+)(c^\lambda_{f,v}).(|0\rangle_{\mathbf{i}} \otimes |0\rangle_{\mathbf{i}}) = |0\rangle_{\mathbf{i}} \otimes |\mathbf{m}\rangle_{\mathbf{i}}$ by Theorem 5.32. Then,

$$\Gamma_w(|0\rangle_{\mathbf{i}} \otimes |\mathbf{m}\rangle_{\mathbf{i}}) = \langle f, v\rangle \Gamma_w(|0\rangle_{\mathbf{i}} \otimes |0\rangle_{\mathbf{i}}) = 0. \tag{5.5}$$

By the properties (5.4), (5.5) and $\Gamma_w(|0\rangle_w \otimes |0\rangle_w) = 1$, we obtain the corollary. □

6. Finite type case

In this section, we assume that $\mathfrak{g}$ is of finite type. Denote by w_0 the longest element of W and by N its length.

6.1. Description of I^{w_0}. We investigate the map I^{w_0}. See Remark-Question 5.33 for the definition of I^{w_0}.

Theorem 6.1. *Let $X \in U^+$, $Y \in U^-$ and $\mathbf{i} \in I(w_0)$. Write $X = \sum_{\mathbf{d} \in \mathbb{Z}^N_{\geq 0}} {}_{\mathbf{i}}\zeta^X_{\mathbf{d}} E^{\mathbf{d}}_{\mathbf{i}}$ and $Y = \sum_{\mathbf{c} \in \mathbb{Z}^N_{\geq 0}} {}_{\mathbf{i}}\zeta^Y_{\mathbf{c}} F^{\mathbf{c}}_{\mathbf{i}}$ (${}_{\mathbf{i}}\zeta^X_{\mathbf{d}}, {}_{\mathbf{i}}\zeta^Y_{\mathbf{c}} \in \mathbb{Q}(q)$).*
For $\lambda \in P_+$, we set

$$I^{w_0}(\tilde{\Omega}(c^\lambda_{(Y.v_\lambda)^*, \omega(X).v_\lambda}).(1 \otimes 1)) = \sum_{\mathbf{c},\mathbf{d} \in \mathbb{Z}^N_{\geq 0}} {}_{\mathbf{i}}\zeta^{\lambda,Y,X}_{\mathbf{c},\mathbf{d}} F^{\mathbf{c}}_{\mathbf{i}} \otimes E^{\mathbf{d}}_{\mathbf{i}}.$$

When $\lambda \in P_+$ tends to ∞ in the sense that $\langle \lambda, \alpha_i^\vee\rangle$ tends to ∞ for all $i \in I$, ${}_{\mathbf{i}}\zeta^{\lambda,Y,X}_{\mathbf{c},\mathbf{d}}$ converges to ${}_{\mathbf{i}}\zeta^Y_{\mathbf{c}}\, {}_{\mathbf{i}}\zeta^X_{\mathbf{d}}$ in the complete discrete valuation field $\mathbb{Q}((q))$ for any $\mathbf{c}, \mathbf{d} \in \mathbb{Z}^N_{\geq 0}$.

Proof. We need only consider the case $X = G^+(\tilde{b}_1^\omega)$ and $Y = G^-(\tilde{b}_2)$ for some $\tilde{b}_1, \tilde{b}_2 \in B(\infty)$.

For sufficiently large $\lambda \in P_+$, we can write $G^-(\tilde{b}_1).v_\lambda = g_{b_1}$ and $G^-(\tilde{b}_2).v_\lambda = g_{b_2}$ for some $b_1, b_2 \in B(\lambda)$. Then,

$$c^\lambda_{g^*_{b_2}, g_{b_1}}.(\langle 0|_{\bar{\mathbf{i}}} \otimes |0\rangle_{\mathbf{i}}) = \sum_{b,b' \in B(\lambda)} (g_b, g_{b'})_\lambda (\pi'^{+}_{w_0}(c^\lambda_{g^*_{b_2}, g^\vee_{b'}}).\langle 0|_{\bar{\mathbf{i}}}) \otimes (\pi^+_{w_0}(c^\lambda_{f_b, g_{b_1}}).|0\rangle_{\mathbf{i}}).$$

It is known that $(g_b, g_{b'})_\lambda \in \mathbb{Z}[q]$ ([4, Proposition 5.1.1]) for any $b, b' \in B(\lambda)$. Hence the theorem is now obtained from Theorem 5.20. □

EXAMPLE 6.2. For $X \in U^+, Y \in U^-$, we can compute the element whose image under the map I^{w_0} is equal to $Y \otimes X$ by Theorem 6.1.

Let $\mathfrak{g} = \mathfrak{sl}_2$. Let us compute the element $\tilde{H}_{2,1} \in U^- \otimes U^+$ such that $I^{w_0}(\tilde{H}_{2,1}) = F^{(2)} \otimes E$. For $n \in \mathbb{Z}_{\geq 0}$, by Proposition 4.17,

$$\begin{aligned}
&\tilde{\Omega}(c^{n\varpi}_{(F^{(2)}.v_{n\varpi})^*, F.v_{n\varpi}}) \\
&= (1-q^{2n})(1-q^{2n-2})\left(\frac{1}{q-q^3} F K_{-(n-2)\varpi} \otimes K_{(n-2)\varpi} + F^{(2)} K_{-n\varpi} \otimes (1-q^{2n}) E K_{n\varpi}\right).
\end{aligned}$$

We have only to consider the terms in $\tilde{\Omega}(c^{n\varpi}_{(F^{(2)}.v_{n\varpi})^*, F.v_{n\varpi}}).(1 \otimes 1)$ which do not depend on n. Hence,

$$\tilde{H}_{2,1} = \frac{1}{q-q^3} F \otimes 1 + F^{(2)} \otimes E.$$

6.2. Some representations of the Drinfeld double $A_q[\mathfrak{g}] \bowtie U'^{\mathrm{cop}}$. Let U' be a variant of the quantized enveloping algebra whose generators of its Cartan part are indexed by the elements of Q (denoted by $\{K_\alpha\}_{\alpha\in Q}$).

Definition 6.3. Define the $\mathbb{Q}(q)$-algebra structure on the $\mathbb{Q}(q)$-vector space $A_q[\mathfrak{g}] \otimes U'$ as follows:

- $A_q[\mathfrak{g}] \to A_q[\mathfrak{g}] \otimes U', F \mapsto F \otimes 1$ and $U' \to A_q[\mathfrak{g}] \otimes U', X \mapsto 1 \otimes X$ is an injective $\mathbb{Q}(q)$-algebra homomorphism,
- $(F \otimes 1)(1 \otimes X) = F \otimes X$ for $F \in A_q[\mathfrak{g}]$ and $X \in U'$,
- $(1 \otimes X)(F \otimes 1) = \sum_{(X)}(X_{(1)}.F.S^{-1}(X_{(3)}))(1 \otimes X_{(2)})$ for $F \in A_q[\mathfrak{g}]$ and $X \in U'$ with $((1 \otimes \Delta) \circ \Delta)(X) = \sum_{(X)} X_{(1)} \otimes X_{(2)} \otimes X_{(3)}$.

This $\mathbb{Q}(q)$-algebra is called the Drinfeld double of $A_q[\mathfrak{g}]$ and U'^{cop}, denoted by $A_q[\mathfrak{g}] \bowtie U'^{\mathrm{cop}}$. (Here "cop" is just a symbol.) We abbreviate $F \otimes X$ to FX for $F \in A_q[\mathfrak{g}]$ and $X \in U'$.

In the rest of this paper, we show that the $A_q[\mathfrak{g}]$-module structure on $\tilde{V}_w$ comes from the $A_q[\mathfrak{g}] \bowtie U'^{\mathrm{cop}}$-module structure in some cases.

Definition 6.4. Define the Hopf algebra automorphism $\mathrm{tw} : \check{U} \to \check{U}$ by $E_i \mapsto -q_i^{-1}E_i$, $F_i \mapsto -q_iF_i$ and $K_\lambda \mapsto K_\lambda$ for $i \in I$ and $\lambda \in P$. Note that $\mathrm{tw} = \omega \circ \psi \circ S$.

The following proposition essentially appears in the reference [12] (under the different convention). We can check it by the direct calculation using Proposition 4.17.

Proposition 6.5. *The map $\tilde{\Omega} \bowtie ((\mathrm{tw} \otimes \mathrm{id}) \circ \Delta \circ \omega) : A_q[\mathfrak{g}] \bowtie U'^{\mathrm{cop}} \mapsto \check{U} \otimes \check{U}, F \otimes X \mapsto \tilde{\Omega}(F)(\mathrm{tw} \otimes \mathrm{id})(\Delta(\omega(X)))$ ($F \in A_q[\mathfrak{g}], X \in U'$) is an injective $\mathbb{Q}(q)$-algebra homomorphism.*

Lemma 6.6. *Let J be a subset of I and W_J the subgroup of W generated by $\{s_j\}_{j\in J}$. Write the longest element of W_J as $w_{J,0}$. Then,*

$$\sum_{\substack{\tilde{b}\in B(\infty) \\ \varepsilon_j^*(\tilde{b})=0 \text{ for all } j\in J}} \mathbb{Q}(q)G^-(\tilde{b})^\vee = \bigcap_{j\in J} \mathrm{Ker}\, {}_je' = U^-(w_0w_{J,0}).$$

Proof. The first equality follows from the equality

$$\sum\nolimits_{\tilde{b}\in B(\infty), \varepsilon_i^*(\tilde{b})=0} \mathbb{Q}(q)G^-(\tilde{b})^\vee = \mathrm{Ker}\, {}_ie'$$

for $i \in I$. Let us denote by U_J^- the $\mathbb{Q}(q)$-subalgebra of U^- generated by $\{F_j\}_{j\in J}$ and Set $(U_J^-)_+ := U_J^- \cap \mathrm{Ker}\,\varepsilon$. Recall that ε is the counit of U. Then, the second equality follows from

$$\bigcap\nolimits_{j\in J} \mathrm{Ker}\, {}_je' = \{Y \in U^- \mid (Y, U^-(U_J^-)_+)_{\mathrm{neg}} = 0\},$$

because the left-hand side includes $U^-(w_0w_{J,0})$ by Proposition 5.3 and the dimension of each weight space of the right-hand side coincides with that of $U^-(w_0w_{J,0})$ by the existence of PBW-bases and $w_{J,0}(\Delta^- \setminus \Delta_J^-) = \Delta^- \setminus \Delta_J^-$ (Δ^- and Δ_J^- are the set of negative roots of $\mathfrak{g}$ and $\mathfrak{g}((a_{jk})_{j,k\in J})$ respectively). □

Let $J \subset I$. By Lemma 6.6, we have

$$U^-(w_0w_{J,0})^\perp = \textstyle\sum_{j\in J} U^-F_j \text{ and } U^+(w_0w_{J,0})^\perp = \sum_{j\in J} U^+E_j.$$

Write $X^+ = E$ and $X^- = F$. Let $M(0)^-$ (resp. $M(0)^+$) be a Verma $\check{U}$-module with highest (resp. lowest) weight 0. (i.e. $M(0)^\mp := \check{U}/\sum_{i\in I}\check{U}X_i^\pm + \sum_{\lambda\in P}\check{U}(K_\lambda - 1)$.) The fixed highest (resp. lowest) weight vector is denoted by m^- (resp. m^+).

Then $\sum_{j\in J} U^\mp X_j^\mp.m^\mp$ is a $\check{U}$-submodule of $M(0)^\mp$ and

$$U^\mp/U^\mp(w_0w_{J,0})^\perp \to M(0)^\mp/\sum_{j\in J} U^\mp X_j^\mp.m^\mp, [1]_{w_0w_{J,0}} \mapsto m^\mp \bmod \sum_{j\in J} U^\mp X_j^\mp.m^\mp$$

gives an isomorphism of $\check{U}^{\lesseqgtr 0}$-modules. Therefore the $\check{U}^{\lesseqgtr 0}$-module structure on $U^\mp/U^\mp(w_0w_{J,0})^\perp$ gives rise to the $\check{U}$-module structure. Combining this fact with Theorem 5.32 and Proposition 6.5, we obtain the following theorem:

Theorem 6.7. *Let J be a subset of I. Then the $A_q[\mathfrak{g}]$-module structure on $\tilde{V}_{w_0w_{J,0}}$ can be extended to the $A_q[\mathfrak{g}] \bowtie U'^{\mathrm{cop}}$-module structure.*

Acknowledgements. The author is greatly indebted to Yoshiyuki Kimura for his significant comments on the first draft of this paper, and to Yuki Arano for discussions concerning Subsection 6.2. He appreciates the supervision of Yoshihisa Saito. He also would like to thank Ryo Sato and Toshiyuki Tanisaki for many helpful discussions and comments. This work was supported by Grant-in-Aid for JSPS Fellows (No. 15J09231) and the Program for Leading Graduate Schools, MEXT, Japan.

References

[1] A. Berenstein and A. Zelevinsky: *Quantum cluster algebras*, Adv. Math. **195** (2005), 405–455.

[2] K.R. Goodearl and M.T. Yakimov: *The Berenstein-Zelevinsky quantum cluster algebra conjecture*, arXiv:1602.00498.

[3] A. Joseph: Quantum groups and their primitive ideals, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, Bd. **29**, Springer-Verlag, Berlin, 1995.

[4] M. Kashiwara: *On crystal bases of the Q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516.

[5] M. Kashiwara: *The crystal base and Littelmann's refined Demazure character formula*, Duke Math. J. **71** (1993), 839–858.

[6] M. Kashiwara: *Global crystal bases of quantum groups*, Duke Math. J. **69** (1993), 455–485.

[7] M. Kashiwara: *Crystal bases of modified quantized enveloping algebra*, Duke Math. J. **73** (1994), 383–413.

[8] M. Kashiwara: *On level-zero representations of quantized affine algebras*, Duke Math. J. **112** (2002), 117–175.

[9] S. Kato: *Poincaré-Birkhoff-Witt bases and Khovanov-Lauda-Rouquier algebras*, Duke Math. J. **163** (2014), 619–663.

[10] Y. Kimura: *Remarks on quantum unipotent subgroups and the dual canonical basis*, Pacific J. Math. **286** (2017), 125–151.

[11] L.I. Korogodski and Y.S. Soibelman: Algebras of functions on quantum groups. Mathematical Surveys and Monographs **56**, Part I, American Mathematical Society, Providence, RI, 1998.

[12] U. Krähmer: *FRT-duals as quantum enveloping algebras*, J. Algebra **264** (2003), 68–81.

[13] A. Kuniba, M. Okado and Y. Yamada: *A common structure in PBW bases of the nilpotent subalgebra of $U_q(\mathfrak{g})$ and quantized algebra of functions*, SIGMA **9** (2013), 049, 23 pages.

[14] G. Lusztig: *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447–498.
[15] G. Lusztig: *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), 365–421.
[16] G. Lusztig: *Canonical bases in tensor products*, Proc. Nat. Acad. Sci. U.S.A. **89** (1992), 8177–8179.
[17] G. Lusztig: *Braid group action and canonical bases*, Adv. Math. **122** (1996), 237–261.
[18] G. Lusztig: Introduction to quantum groups, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2010, Reprint of the 1994 edition.
[19] P.J. McNamara: *Finite dimensional representations of Khovanov-Lauda-Rouquier algebras I: Finite type*, J. Reine Angew. Math. **707** (2015), 103–124.
[20] P.J. McNamara: *Representations of Khovanov-Lauda-Rouquier algebras III: symmetric affine type*, Math. Z. **287** (2017), 243–286.
[21] B. Narayanan: *Representations of affine quantum function algebras*, J. Algebra **272** (2004), 775–800.
[22] H. Oya: *Representations of quantized function algebras and the transition matrices from canonical bases to PBW bases*, arXiv:1501.01416.
[23] Y. Saito: *Quantized coordinate rings, PBW-type bases and q-boson algebras*, J. Algebra **453** (2016), 456–491.
[24] Y. Saito: *PBW basis of quantized universal enveloping algebras*, Publ. Res. Inst. Math. Sci. **30** (1994), 209–232.
[25] Y.S. Soibelman: *Algebra of functions on a compact quantum group and its representations*, Algebra i Analiz. **2** (1990), 190–212.
[26] T. Tanisaki: *Modules over quantized coordinate algebras and PBW-bases*, J. Math. Soc. Japan **69** (2017), 1105–1156.
[27] L.L. Vaksman and Y.S. Soibelman: *An algebra of functions on the quantum group* SU(2), Funktsional. Anal. i Prilozhen. **22** (1988), 1–14, 96.

Graduate School of Mathematical Sciences
The University of Tokyo
Komaba, Tokyo, 153–8914
Japan
[*Current Address*:]
Sorbonne Paris Cité Université Paris Diderot
CNRS Institut de Mathématiques de Jussieu
Paris Rive Gauche UMR7586
Bâtiment Sophie German
Case 7012, 75205, Paris Cedex 13
FRANCE
e-mail: hironori.oya@imj-prg.fr